Zermelo-Fraenkel Set Theory

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In mathematics, a **set** is a collection of elements such as the numbers $\{1, 2, 3\}$ or the natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$. Mathematical sets have certain well-known properties: for example, we can take the union of two sets, or form the Cartesian product of two sets, or define a function from one set to another one. Sets, products, and functions are fundamental concepts in mathematics.

A **set theory** is a formal specification of what sets exist, and what operations on sets are allowed. A well-specified set theory is essential for ensuring that mathematical reasoning is valid, e.g., that it is free from self-contradictions and can be automated according to generally accepted rules of logical inference.

There are at least three reasons for studying set theory:

1. It provides insight into the foundations of mathematics.
2. It provides a basis for studying subjects such as category theory that are built on set theory.
3. It is a deep and interesting area of mathematics in its own right.

This paper gives a brief introduction to a formal set theory called **Zermelo-Fraenkel set theory**, or ZF. ZF was developed by Ernst Zermelo, Abraham Fraenkel, and others in the first part of the twentieth century. ZF is usually augmented with an axiom called the **axiom of choice**. In this form it is called Zermelo-Fraenkel set theory with the axiom of choice, or ZFC. ZFC is the primary set theory in use by mathematicians today.

The paper assumes some familiarity with first-order logic, but otherwise it develops everything from first principles. The paper proceeds as follows. In § 1, we describe the formal language that we will use for ZFC. In § 2, we extend the formal language with so-called "syntactic sugar" that makes the logical formulas easier to read and write, but doesn't add any expressive power. In § 3, we state the **axioms** of ZFC. These are the formulas, stated in the extended language, that we assume to be true and that we can use to derive other true statements. In § 4, we give examples that show how to use the formal theory to construct proofs. In § 5, we briefly discuss some next steps in studying set theory.

1. Formal Language

In this section we describe the formal language that we will use to state axioms and construct proofs in ZFC. The language is based on the standard language of first-order logic. First we give the syntax (the form of valid statements in the language) and then we give the semantics (the meaning of valid statements).

Syntax: The language syntax consists of the following elements:

1. **Terms:** A term t is one of the following:
   a. A **value** a, b, or c.
   b. A **variable** x, y, or z.
2. **Formulas:** A formula p is one of the following:
   a. **Membership:** $t \in t'$.
   b. **Equality:** $t = t'$.
   c. **Universal quantification:** $\forall x\ p$.
   d. **Existential quantification:** $\exists x\ p$.
   e. **One-way implication:** $(p \Rightarrow p')$.
   f. **Two-way implication:** $(p \Leftrightarrow p')$.

g. **Conjunction:** $(p \wedge p')$.

h. **Disjunction:** $(p \vee p')$.

i. **Negation:** $\neg p$.

The definition for formulas is recursive. For example, the definition of universal quantification says, "If p is a formula, then $\forall x\ p$ is a formula." The non-recursive cases are membership and equality.

The parentheses around the binary formulas ensure that the grammar is unambiguous, without requiring any special rules for precedence and associativity. For example:

- $\forall x \exists y (p \Rightarrow p')$ says, "For all x, there exists y such that if p then p'."
- $\forall x (\exists y\ p \Rightarrow p')$ says, "For all x, if there exists y such that p, then p'."

Semantics: Terms t represent sets.[1] A value a, b, or c is a set. A variable x, y, or z is a symbol that stands in for a set in a quantified formula.

Formulas p are either true or false. We write $p[x \leftarrow a]$ to denote the formula formed by substituting the value a for the variable x in p. For example, let p be the formula $x = y$. Then $p[x \leftarrow a]$ is the formula $a = y$, and $p[x \leftarrow a][y \leftarrow b]$ is the formula $a = b$.

To simplify the substitution rules, we assume that no formula quantifying over a variable x contains a subformula that quantifies over the same variable x. For example, we never have $\forall x \exists x\ p$. Examples of permitted formulas are $\forall x \exists x'\ p$ and $(\forall x\ p \wedge \forall x\ p')$. We will call a formula p **well-formed** if it conforms to the syntactic rules for formulas and it satisfies this assumption about variable names. We can always satisfy the assumption by renaming variables if necessary.

The formulas have the following meanings:

1. $t \in t'$ means that the set represented by t is a member of the set represented by t'.
2. $t = t'$ means that the set represented by t is equal to the set represented by t'.
3. $\forall x\ p$ means that $p[x \leftarrow a]$ is true for every set a.
4. $\exists x\ p$ means that $p[x \leftarrow a]$ is true for some set a.
5. $(p \Rightarrow p')$ means that if p is true, then p' is true.
6. $(p \Leftrightarrow p')$ means that p' is true if and only if p is true.
7. $(p \wedge p')$ means that p is true and p' is true. (The $\wedge$ symbol looks like an A, which stands for "and.")
8. $(p \vee p')$ means that at least one of p and p' is true. (The $\vee$ symbol looks like a V, which stands for "vel," which is "or" in Latin.)
9. $\neg p$ means that p is false.

2. Extended Syntax

To assist in reading and writing ZFC formulas, we extend the syntax of the formal language. We use a technique called **syntactic sugar**: for each element E of extended syntax, we explain how to transform E into a corresponding element of the formal syntax described in the previous section. This syntactic transformation is called **desugaring**. It ensures that the extended syntax adds nothing to the formal language except notational convenience.[2]

Terms: We add the following elements to the syntax of terms t:

1. **Empty set:** We add the term $\varnothing$ representing the unique set with no members.
2. **Single-member set:** We add the term $\{t\}$ representing the set with t as its only member.
3. **Pair set:** We add the term $\{t, t'\}$ representing the set with t and t' as its only members.

[1] For an alternate formalism that includes individuals (i.e., set elements that are not themselves sets), see [Suppes 1972]. We use the set-only formalism here because it is more straightforward and more standard. We can always represent individuals as sets, so we don't lose any power this way.

[2] See [Suppes 1972] for an alternate approach that uses logical inference. In this approach, each new bit of syntax has an associated axiom of definition that allows the formal reasoning system to "infer away" the added syntax. In this approach, one has to prove that the axioms of definition do not add any deductive power to the set theory axioms.

4. **Ordered pair:** We add the term (t, t') representing the ordered pair with t first and t' second.
5. **Set union:** We add the term $(t \cup t')$ representing the union of sets t and t', i.e., the set of all elements each of which is a member of t or t'.
6. **Set intersection:** We add the term $(t \cap t')$ representing the intersection of the sets t and t', i.e., the set of all elements each of which is a member of t and t'.
7. **Function evaluation:** We add the term $t(t')$ representing the evaluation of function t at value t'. As usual, we represent a function f as a set of ordered pairs $(a, f(a))$. Below we explain how to represent ordered pairs.

To desugar a term t, we desugar the smallest formula p enclosing t. For example, to desugar the term $\varnothing$ in the formula $(p \wedge x = \varnothing)$, we desugar the formula $x = \varnothing$.

We may represent any formula p containing a term t as $p'[y \leftarrow t]$, where p' contains a free (unquantified) occurrence of y, y does not appear in t or p, and $p'[y \leftarrow t]$ denotes the formula formed by substituting the term t for the variable y. For example, we may represent the formula $x = \varnothing$ as $x = y\ [y \leftarrow \varnothing]$. In most cases, we desugar $p'[y \leftarrow t]$ to $\forall y(p'' \Rightarrow p')$, where p'' constrains y according to the meaning of t.

Here are the rules for desugaring terms embedded in formulas:

1. **Empty set:** We desugar $p[y \leftarrow \varnothing]$ to $\forall y(\forall y'\ y' \notin y \Rightarrow p)$.
2. **Single-member set:** We desugar $p[y \leftarrow \{t\}]$ to $\forall y(\forall y'(y' \in y \Leftrightarrow y' = t) \Rightarrow p)$.
3. **Pair set:** We desugar $p[y \leftarrow \{t, t'\}]$ to $\forall y(\forall y'(y' \in y \Leftrightarrow (y' = t \vee y' = t')) \Rightarrow p)$.
4. **Ordered pair:** We desugar $p[y \leftarrow (t, t')]$ to $p[y \leftarrow \{\{t\}, \{t, t'\}\}]$. This construction lets us represent ordered pairs as sets, without introducing a new primitive for this purpose. It defines an ordering on the pair t, t' in which the first element t is the one that appears in a single-member set.
5. **Set union:** We desugar $p[y \leftarrow (t \cup t')]$ to $\forall y(\forall y'(y' \in y \Leftrightarrow (y' \in t \vee y' \in t')) \Rightarrow p)$.
6. **Set intersection:** We desugar $p[y \leftarrow (t \cap t')]$ to $\forall y(\forall y'(y' \in y \Leftrightarrow (y' \in t \wedge y' \in t')) \Rightarrow p)$.
7. **Function evaluation:** We desugar $p[y \leftarrow t(t')]$ to $((\mathsf{fun}(t) \wedge \forall y\ (t', y) \in t) \Rightarrow p)$. See below for the desugaring of the formula $\mathsf{fun}(t)$.

Desugaring the smallest enclosing formula ensures that each use of a variable occurs in the scope were it is quantified. For example, to desugar the formula

$$\forall x \exists x'\ (x \cup x') = x,$$

we rewrite the formula inside the quantifiers as $y = x\ [y \leftarrow (x \cup x')]$ and desugar it to obtain

$$\forall x \exists x' \forall y(\forall y'(y' \in y \Leftrightarrow (y' \in x \vee y' \in x')) \Rightarrow y = x).$$

We have constructed the desugaring rules so that for each desugaring $p'[y \leftarrow t]$ to $\forall y(p'' \Rightarrow p')$, there is a unique set a such that $p''[y \leftarrow a]$ is true. This fact ensures that each term t uniquely refers to a set. In §4, we show how to prove this fact for the term $\varnothing$. In this case, a is the empty set, i.e., the unique set with no members. The other proofs are similar. All the proofs rely on the axioms that we will state in § 3.

Formulas: We add the following elements to the syntax of formulas p:

1. **Negated membership:** We add the formula $t \notin t'$ stating that t is not a member of t'. We desugar it to $\neg t \in t'$.
2. **Negated equality:** We add the formula $t \neq t'$ stating that t is not equal to t. We desugar it to $\neg t = t'$.
3. **Subset relation:** We add the formula $t \subseteq t'$ stating that t is a subset of t'. We desugar it to

   $$\forall x(x \in t \Rightarrow x \in t'),$$

   where x does not appear in t or t'.
4. **Relations:** We add the formula $\mathsf{rel}(t)$ stating that t represents a relation on a pair of sets. We desugar it to

   $$\forall x(x \in t \Rightarrow \exists y \exists z\ x = (y, z)),$$

   where none of x, y, and z appears in t. This definition says that t represents a set of ordered pairs.

5. **Functions:** We add the formula $\mathsf{fun}(t)$ stating that t represents a function from one set to another. We desugar it to

$$(\mathsf{rel}(t) \wedge \forall x \forall y \forall y'(((x, y) \in t \wedge (x, y') \in t) \Rightarrow y = y')),$$

where none of x, y, and z appears in t. This definition says that t represents a relation that is also well-defined as a function.

6. **Unique existential quantification:** We add the formula $\exists ! x\ p$ stating that there exists one and only one set a such that $p[x \leftarrow a]$ is true. We desugar it to

$$\exists x(p \wedge \forall x'(p[x \leftarrow x'] \Rightarrow x' = x)),$$

where x' does not appear in p.

7. **Truth:** We add the formula true that is always true. We desugar it to $\forall x\ x = x$.
8. **Falsity:** We add the formula false that is always false. We desugar it to $\neg\mathsf{true}$.

3. Axioms

In this section we state the axioms of ZFC. Each axiom is a formula in the extended language defined in § 2. After desugaring, each axiom is by assumption a true formula in the formal language (§ 1). Any formula that can be derived from the axioms via the rules of logical inference is also a true formula.

Axiom of Extensionality: In informal mathematical language, the Axiom of Extensionality states that any two sets a and a' are equal if they have the same elements, i.e., any set b is a member of both or neither. In the extended formal language:

$$\forall x \forall x'(\forall y(y \in x \Leftrightarrow y \in x') \Rightarrow x = x')$$

Rationale: This axiom provides the fundamental notion of equality of two sets. The converse of this axiom is true by the substitution property of equality in logic. For example, if $b \in x\ [x \leftarrow a]$ is true and $a = a'$ is true, then $b \in x\ [x \leftarrow a']$ is true. Therefore if $b \in a$ is true and $a = a'$ is true, then $b \in a'$ is true.

Axiom of Pairing: Informally, the Axiom of Pairing states that for any two sets a and a', the set $\{a, a'\}$ exists, i.e., there exists a set that has exactly a and a' as elements. In the extended formal language:

$$\forall x \forall x' \exists y \forall z(z \in y \Leftrightarrow (z = x \vee z = x'))$$

Rationale: This axiom provides a basic set construction, using a set as a member of another set. In particular, it provides the construction of $\{a\} = \{a, a\}$.

Axiom of Union: Informally, the Axiom of Union states that for any set a, the union $\bigcup_{c \in a} c$ over the members of a exists, i.e., there exists a set b whose members are exactly the sets c', each of which is a member of some member c of a. In the extended formal language:

$$\forall x \exists y \forall z'(z' \in y \Leftrightarrow \exists z(z' \in z \wedge z \in x))$$

Rationale: This axiom provides a basic set construction, taking the union of a collection of sets. Note that with this axiom and the axiom of pairing, we can construct any set consisting of a finite number of sets. For example, in informal mathematical notation, we have

$$\{a, b, c\} = \cup\ \{\{a, b\}, \{c\}\}.$$

Axiom of Power Set: Informally, the Axiom of Power Set states that for any set a, the power set of a exists, i.e., there exists a set b whose members are exactly the subsets c of a, including the empty set $\varnothing$ and the set a itself. In the extended formal language:

$$\forall x \exists y \forall z(z \in y \Leftrightarrow z \subseteq x)$$

Rationale: This axiom provides a basic set construction, taking the power set of a set.

Axiom Schema of Specification: Let p be a formula in the extended formal language such that $\forall x\ p$ is a well-formed formula (i.e., p is syntactically well-formed, and x may appear free in p, but no subformula of p quantifies over x). Informally, the Axiom Schema of Specification states that for every set a, the set $\{c \in a\colon p[x \leftarrow c]\}$ exists, i.e., there exists a set b whose members are exactly the members c of a such that $p[x \leftarrow c]$ is true. In the extended formal language:

$$\forall x \exists y \forall z (z \in y \Leftrightarrow (z \in x \wedge p)),$$

where neither y nor z appears in p. This statement is called an **axiom schema** because there is one axiom for each formula p.

Rationale: This axiom provides a basic construction, set comprehension. By restricting the comprehension to elements in a base set a, the axiom avoids paradoxical constructions such as the set of all sets that do not contain themselves.

Axiom Schema of Replacement: Let p be a formula in the extended formal language such that $\forall y \exists y'\ p$ is a well-formed formula. Informally, we say that p is a **function formula** f on a set a if, for each $b \in a$, there is a unique set b' such that $p[y \leftarrow b][y' \leftarrow b']$ is true. In this case, in our informal mathematical language we write $f(b) = b'$. The prototypical example of a function formula is $t(y) = y'$, where t is a function as defined in § 2.

The Axiom Schema of Replacement says that if p is a function formula f on a, then there exists a set c whose members are exactly the sets $f(b)$ for $b \in a$. We call c the **image** of f and write $f(a) = c$. In the extended formal language:

$$\forall x (\forall y (y \in x \Rightarrow \exists ! y'\ p) \Rightarrow \exists z \forall y (y \in x \Leftrightarrow \exists y' (y' \in z \wedge p))),$$

where x does not appear in p.

Rationale: This axiom ensures that the image of a function is a set.

Axiom of Regularity: Informally, the Axiom of Regularity says that every nonempty set a has a member b such that a and b are disjoint sets (i.e., have empty intersection). In the extended formal language:

$$\forall x (x \neq \varnothing \Rightarrow \exists y (y \in x \wedge (y \cap x) = \varnothing))$$

Rationale: This axiom excludes various problematic constructions that would otherwise be possible. For example, together with the Axiom of Pairing, it ensures that no set is an element of itself (i.e., $\forall x\ x \notin x$). See § 4.3.

Axiom of Infinity: Informally, the Axiom of Infinity says that there exists a set a that contains the empty set $\varnothing$, and such that for each member $b \in a$, we have $(b \cup \{b\}) \in a$. In the extended formal language:

$$\exists x (\varnothing \in x \wedge \forall y (y \in x \Rightarrow (y \cup \{y\}) \in x))$$

We may list the first few elements of the set a guaranteed by this axiom as follows:

$$a = \{a_0, a_1, a_2, \ldots\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \ldots\}$$

In general we have $a_0 = \varnothing$ and $a_i = \{a_0, \ldots, a_{i-1}\}$ for $i > 0$.

Rationale: Together with the Axiom of Specification, this axiom guarantees that the empty set exists. See § 4.1. This axiom also guarantees the existence of the set a shown above, which has infinitely many elements. Indeed, for all $j \geq i \geq 0$, we have $a_i \subseteq a_j$. Further, the Axiom of Regularity guarantees that for all y, $y \notin y$. Therefore for all $j > i \geq 0$, a_j contains an element that is not in any of the a_i, and so a_j is not equal to any of the a_i.

Axiom of Choice: Informally, the Axiom of Choice says that for any set a of nonempty sets, there is a "choice function" b defined on a such that for all c in a, b chooses an element of c, i.e., $b(c) \in c$. A famous example given by Bertrand Russell is a choice function on a set of pairs of socks that chooses, from each pair, either the left or the right sock. In the extended formal language:

$$\forall x (\varnothing \notin x \Rightarrow \exists y \forall z (z \in x \Rightarrow y(z) \in z))$$

Rationale: This axiom, though non-constructive, is generally accepted. Many standard mathematical results require this axiom for their proofs.

4. Example Proofs

In this section, we give some examples of using the formal language of ZFC to carry out proofs, i.e., to derive true formulas from the axioms, previously derived true formulas, and the rules of logic. We describe the proofs informally. It is not hard to write these proofs in a fully formal and machine-checkable way. However, that is best done with the assistance of a computer.

4.1. Proof That the Empty Set Exists

Let p' be the formula of the Axiom Schema of Specification after setting p to false. From p' and the standard logical rules for conjunction and implication, we can derive

$$\forall x \exists y \forall z \; z \notin y. \tag{1}$$

By the Axiom of Infinity, we have

$$\exists x \;\; p'', \tag{2}$$

where p'' is the formula stated in the axiom. By the rules of logic, (1) and (2) yield

$$\exists x (p'' \wedge \exists y \forall z \; z \notin y). \tag{3}$$

By the rules of conjunction, we can eliminate p'' from (3). Then the resulting formula does not refer to x, so we can eliminate the quantification over x. This yields

$$\exists y \forall z \; z \notin y,$$

which is the result we wanted.

4.2. Proof That the Empty Set Is Unique

Assume the existence of sets a and a' such that the following formula is true:

$$(\forall y \; y \notin a \wedge \forall y \; y \notin a') \tag{4}$$

Rearranging terms gives

$$\forall y (y \notin a \wedge y \notin a') \tag{5}$$

Then (5) implies

$$\forall y (y \in a \Leftrightarrow y \in a') \tag{6}$$

By the Axiom of Extensionality with $x = a$ and $x' = a'$, we have

$$\forall y (y \in a \Leftrightarrow y \in a') \Rightarrow a = a' \tag{7}$$

Putting (6) together with (7) yields $a = a'$, q.e.d.

4.3. Proof That No Set Contains Itself

Fix a set a. Sections 4.1 and 4.2 establish the truth of the formula

$$\exists ! x \;\; x = \emptyset \tag{8}$$

A similar argument using the Axiom of Pairing establishes

$$\exists ! x \;\; x = \{a\} \tag{9}$$

(Try writing out this argument as an exercise.) Let b be the unique set guaranteed by formula (8), and let Let c be the unique set guaranteed by formula (9). Then $a \in c$ and $a \notin b$, so by the converse of the Axiom of Extensionality, $b \neq c$ and so $c \neq \emptyset$. Therefore by the Axiom of Regularity, we have

$$\exists y (y \in c \wedge (y \cap c) = \emptyset) \tag{10}$$

The only element of $c = \{a\}$ is a, so (10) implies

$$a \cap c = \emptyset \tag{11}$$

Because $a \in c$, (11) implies $a \notin a$, q.e.d.

4.4. Proof That the Cartesian Product of Sets Exists

Fix sets a and b. We wish to prove the existence of a set c whose members are precisely the elements (a', b') with $a' \in a$ and $b' \in b$. Here $(a', b') = \{\{a'\}, \{a', b'\}\}$ is the ordered pair construction introduced in § 2. Informally, we write $c = a \times b$ and call c the **Cartesian product** of the sets a and b.

Formally, we want to show

$$\exists z \forall z'(z' \in z \Leftrightarrow p), \tag{12}$$

where p is the formula

$$\exists x \exists y((x \in a \wedge y \in b) \wedge z' = (x, y)). \tag{13}$$

Let c' be the power set of $a \cup b$. This set exists by the Axiom of Power Set and because

- The Axiom of Pairing ensures that the set $\{a, b\}$ exists.
- The Axiom of Union ensures that the set $\cup\{a, b\} = a \cup b$ exists.

c' contains all (but not only) sets of the form $\{a'\}$ and $\{a', b'\}$ with $a' \in a$ and $b' \in b$. Now let c'' be the power set of c'. This set exists by the Axiom of Power Set. It contains all (but not only) sets of the form $\{\{a'\}, \{a', b'\}\}$ with $a' \in a$ and $b' \in b$. Therefore we have

$$\forall z'(p \Leftrightarrow (z' \in c'' \wedge p)) \tag{14}$$

By (14) it suffices to show

$$\exists z \forall z'(z' \in z \Leftrightarrow (z' \in c'' \wedge p)) \tag{15}$$

But (15) follows immediately from the Axiom Schema of Specification.

5. Next Steps

This document describes only the very beginning of set theory. There is much more. For example, one can use set theory to

- Develop a theory of **cardinal numbers** used for counting and arithmetic.
- Develop a theory of **ordinal numbers** used for ordering sets. (The **finite ordinals** are just the sets guaranteed by the Axiom of Infinity.)
- Develop a theory of the real numbers.

For details, see, e.g., [Suppes 1972].

References

Suppes, Patrick. *Axiomatic Set Theory.* Dover Publications, Inc., 1972.