The Riemann-Roch Theorem

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The **Riemann-Roch theorem** is a fundamental result in the theory of Riemann Surfaces and in the related area of complex algebraic curves. Given a compact Riemann surface R and an object D called a **divisor** that constrains the zeros and poles of a meromorphic function on R, the Riemann-Roch theorem describes the space $L(D)$ of meromorphic functions on R such that each function in $L(D)$ satisfies the constraints implied by D.

This paper develops the essential theory necessary to state and prove the Riemann-Roch theorem for compact Riemann surfaces. The full theory is available, for example, in [Miranda 1995] and in the references cited therein. For a more elementary approach based on complex algebraic curves, see [Kirwan 1992]. For a related result, also called the Riemann-Roch theorem, in the area of algebraic curves over algebraically closed fields, see [Fulton 2008]. In this version, meromorphic functions on Riemann surfaces are replaced by rational functions on algebraic curves.

This paper assumes that you are familiar with the material covered in my papers *Complex Charts on Topological Surfaces* and *Holomorphic Maps Between Riemann Surfaces*. It also assumes some knowledge of groups, rings, vector spaces, and homomorphisms between these objects. These topics are covered in my paper *Definitions for Commutative Algebra*.

1. Meromorphic One Forms

Let $R = (T, A)$ be a Riemann surface, where the atlas A is a family of charts $\{C_i = (U_i, \phi_i)\}$. Recall that a holomorphic one form ω on R is a family of holomorphic complex one forms $\{\omega_i = f_i \; dz\}$, one on the open set $\phi_i(U_i) \subseteq \mathbf{C}$ for each chart C_i, such that for any two charts C_i and C_j we have

$$\omega_i = \phi_{ij}^* \; \omega_j$$

on $\phi_i(U_i \cap U_j)$. Here ϕ_{ij} is the transition function $\phi_j \circ \phi_i^{-1}$, and $\phi_{ij}^* \; \omega_j$ is the pullback of ω_j with respect to ϕ_{ij}, i.e., the complex one form defined by

$$(\phi_{ij}^* \; \omega_j)(z) = \omega_j(\phi_{ij}(z)) \circ d\phi_{ij}(z) = f_j(\phi_{ij}(z))\phi'_{ij}(z) \; dz.$$

See *Complex Charts on Topological Surfaces*, § 2.3 and § 4.3.

A **meromorphic one form** is identical to a holomorphic one form, except that we allow the complex one forms $\omega_i = f_i \; dz$ to be meromorphic. That is, each complex function f_i is holomorphic on $\phi_i(U_i) - P_i$, where P_i is a discrete (and possibly empty) set of poles; and at each point in P_i, f_i has a Laurent series expansion with a positive and finite number of negative powers.

In this section, we establish some basic facts about meromorphic one forms.

1.1. An Example

We begin with an example. Let $R = (T, A)$ be the Riemann sphere $\mathbf{C}_\infty$, where A is the maximal atlas containing the standard charts $C_1 = (U_1, \phi_1)$ and $C_2 = (U_2, \phi_2)$ with domains $U_1 = \mathbf{C}$ and $U_2 = \mathbf{C} - \{0\} \cup \{\infty\}$. Define $\omega_1 = z \; dz$ and $\omega_2 = -(1/z^3) \; dz$. Then $\phi_{12} = \phi_{21} = 1/z$, $\phi'_{12} = \phi'_{21} = -z^{-2}$, and by the definition of the pullback we have $\omega_2 = \phi_{21}^* \; \omega_1$ and $\omega_1 = \phi_{12}^* \; \omega_2$.

Let $C_i = (U_i, \phi_i)$ be any chart of $\mathbf{C}_\infty$. Then we have $U_i \subseteq U_\alpha$ for exactly one of $\alpha = 1$ and $\alpha = 2$, and we may define $\omega_i = \phi_{i\alpha}^* \omega_\alpha$. We will now show that this definition yields a valid meromorphic one form on $\mathbf{C}_\infty$, i.e., for all charts C_i and C_j of A, we have $\phi_i = \phi_{ij}^* \omega_j$.

Recall from *Complex Charts on Topological Surfaces*, § 2.3, that if g and h are holomorphic complex functions and ω is a holomorphic complex one form, then we have

$$g^*(h^*\omega) = (h \circ g)^*\omega. \tag{1}$$

The same proof goes through for meromorphic complex one forms. Equation (1) is very useful; henceforth we will call it the **pullback composition lemma** for meromorphic one forms.

Let C_i and C_j be any two charts of A. Assume that U_i is contained in U_α and U_j is contained in U_β, where each of α and β is 1 or 2. Then we have

$$\begin{aligned}
\phi_{ij}^*\omega_j &= \phi_{ij}^*(\phi_{j\beta}^*\omega_\beta) \quad \text{(definition of } \omega_j\text{)} \\
&= (\phi_{j\beta} \circ \phi_{ij})^*\omega_\beta \quad \text{(pullback composition lemma)} \\
&= \phi_{i\beta}^*\omega_\beta \quad \text{(definitions of } \phi_{j\beta}, \phi_{ij}, \text{ and composition)} \\
&= \phi_{i\beta}^*(\phi_{\beta\alpha}^*\omega_\alpha) \quad \text{(shown above for } \alpha \neq \beta\text{; obvious for } \alpha = \beta\text{)} \\
&= (\phi_{\beta\alpha} \circ \phi_{i\beta})^*\omega_\alpha \quad \text{(pullback composition lemma)} \\
&= \phi_{i\alpha}^*\omega_\alpha \quad \text{(definitions of } \phi_{\beta\alpha}, \phi_{i\beta}, \text{ and composition)} \\
&= \omega_i \quad \text{(definition of } \omega_i\text{)},
\end{aligned}$$

which was to be shown.

Note that the proof depends only on the fact that ω_1 and ω_2 are consistent, in the sense that $\omega_\alpha = \phi_{\alpha\beta}^*\omega_\beta$ for α and β in the set $\{1, 2\}$. Thus we have shown that one may give a meromorphic one form on $\mathbf{C}_\infty$ by consistently defining the meromorphic one forms ω_1 and ω_2 on the charts C_1 and C_2. And in fact it suffices to show, for example, that $\omega_1 = \phi_{12}^*\ \omega_2$, for in this case we have

$$\phi_{21}^*\ \omega_1 = \phi_{21}^*(\phi_{12}^*\ \omega_2) = (\phi_{12} \circ \phi_{21})^*\omega_2 = id^*\omega_2 = \omega_2,$$

where id denotes the identity function $z \mapsto z$.

1.2. The Order of a Meromorphic One Form

Let $R = (T, A)$ be a Riemann surface, and let $f\colon R \to \mathbf{C}$ be a meromorphic function. Recall that f has an order at each point p in T, given by

$$\operatorname{ord}_p\ f = \operatorname{ord}_{p_i}\ f_i,$$

where $C_i = (U_i, \phi_i)$ is a chart of A, $p_i = \phi_i(p)$, and f_i is the local function $f \circ \phi_i^{-1}$. The order is well-defined, because it is the same for any choice of chart. See *Holomorphic Maps Between Complex Riemann Surfaces*, § 1.5. We now establish a similar result for meromorphic one forms on R.

Let $R = (T, A)$ be a Riemann surface, and let $\omega = \{\omega_i = f_i\ dz\}$ be a meromorphic one form on R. For each point p in T, define

$$\operatorname{ord}_p\ \omega = \operatorname{ord}_{p_i}\ f_i,$$

where C_i is any chart of A, and $p_i = \phi_i(p)$. We must show that this definition is independent of the choice of chart.

Fix a chart $C_i = (U_i, \phi_i)$ of A that contains p. Use the subset topology in T to make U_i into a topological space T'. Construct an atlas A' on T' such that, for each chart $C'_j = (U'_j, \psi_j)$ in A', U'_j is $U_j \cap U_i$, and ψ_j is the restriction of ϕ_j to U'_j. Then $R' = (T', A')$ is a Riemann surface. Further, since we are interested in the local behavior of R near p, it suffices to show the result for R'. Now remove the primes and replace ψ with ϕ. We have reduced the problem to proving that the order of a meromorphic one form is well-defined at a point p on a Riemann surface $R = (T, A)$ in which $U_j \subseteq U_i$ for some chart C_i containing p and all charts C_j of A.

For each chart C_j in A, define a local function

$$F_j = \phi_{ji}^*\ f_i = f_i \circ \phi_{ji}.$$

By the construction above, F_j is defined on all of $\phi_j(U_j)$. We wish to show that the family $F = \{F_j\}$ is a meromorphic function on R. For this it suffices to show that $F_j = \phi_{jk}^* \, F_k$ for all pairs of charts C_j and C_k in A. See *Complex Charts on Topological Surfaces*, § 2.2 and § 4.2. From the definitions, we have

$$\begin{aligned}\phi_{jk}^* \, F_k &= f_i \circ \phi_{ki} \circ \phi_{jk} \\ &= f_i \circ \phi_i \circ \phi_k^{-1} \circ \phi_k \circ \phi_j^{-1} \\ &= f_i \circ \phi_{ji} \\ &= F_j.\end{aligned}$$

This establishes what we want.

By the previous result for meromorphic functions, we have

$$\mathrm{ord}_{p_j} \, F_j = \mathrm{ord}_{p_k} \, F_k$$

for all pairs of charts C_j and C_k. Because $\phi'_{ji}(p_j) \neq 0$, the power series expansion of ϕ'_{ji} at p_j must have a nonzero constant term. Therefore multiplying by ϕ'_{ji} does not change the order at p_j; and similarly for ϕ'_{ki}. Thus we have

$$\mathrm{ord}_{p_j} \, (F_j \phi'_{ji}) = \mathrm{ord}_{p_k} \, (F_k \phi'_{ki}). \tag{2}$$

Now consider the local one form ω_j. By the definition of the one form ω, we have

$$\omega_j = f_j \, dz = \phi_{ji}^* \, (f_i \, dz) = (f_i \circ \phi_{ji})\phi'_{ji} \, dz = F_j \phi'_{ji} \, dz.$$

Therefore $f_j = F_j \phi'_{ji}$, and similarly for f_k. Together with (2), this fact establishes the result.

1.3. The Pullback of a One Form

Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f \colon R_1 \to R_2$ be a holomorphic map. We say that f has the **chart inclusion property** if, for every chart $C_i = (U_i, \phi_i)$ of R_1, we have a discrete set $P_i \subseteq U_i$ and a chart $D_i = (V_i, \psi_i)$ of R_2 such that $f(U_i - P_i) \subseteq V_i$. In this case we call D_i an **including chart** for C_i with respect to f. Note that it is permitted for P_i to be empty.

For example, let $R = (T, A)$ be a Riemann surface, and let $f \colon R \to \mathbf{C}_\infty$ be a holomorphic map. Then f has the chart inclusion property, because for every chart $C_i = (U_i, \phi_i)$ in A, we can let D_i be the chart $\mathbf{C}_\infty - \{\infty\}$ on $\mathbf{C}_\infty$, and we can let P_i be the points p in U_i such that $f(p) = \infty$. Then P_i is a discrete set, and $f(U_i - P_i)$ lies in D_i, as required.

Let $\omega = \{g_i \, dz\}$ be a meromorphic one form on R_2. If f has the chart inclusion property, then we can define a meromorphic one form on R_1, called a **pullback** of ω via f and written $f^*\omega$, as follows. For each chart C_i in R_1, choose an including chart D_i in R_2. Let h_i be the holomorphic function $\psi_i \circ f \circ \phi_i^{-1}$. Let χ denote $f^*\omega$. On $\phi_i(U_i - P_i)$, define

$$\boxed{\chi_i = (f^*\omega)_i = h_i^* \omega_i = (g_i \circ h_i) h'_i \, dz.} \tag{3}$$

At each point p of $\phi_i(P_i)$, if $(g_i \circ h_i) h'_i$ has a removable singularity, then we remove the singularity in the definition of χ at p. Otherwise χ has a pole at p.

For each chart C_i in R_1, we call the chart D_i in the definition of $f^*\omega$ the **corresponding chart** to C_i with respect to the pullback $f^*\omega$. Note that the pullback $f^*\omega$ is not uniquely determined by f and by ω; it also depends on the choice of corresponding charts. We will adopt the convention that $f^*\omega$ denotes any pullback of ω via f.

We must show that equation (3) defines a valid one form on R_1, i.e., for all pairs of charts $C_i = (U_i, \phi_i)$ and $C_j = (U_j, \phi_i)$ on R_1, we have $\chi_i = \phi_{ij}^* \, \chi_j$. Because ω is a one form, on $\phi_i(U_i - P_i)$ we have

$$\chi_j = h_j^* \omega_j = h_j^* (\psi_{ji}^* \omega_i).$$

By the pullback composition lemma (§ 1.1), we have

$$\chi_j = (\psi_{ji} \circ h_j)^* \omega_i.$$

By the definition of ψ_{ji} and of h_j, this yields

$$\chi_j = (\psi_i \circ f \circ \phi_j^{-1})^* \omega_i.$$

By the pullback composition lemma again, we have

$$\phi_{ij}^* \chi_j = \phi_{ij}^* (\psi_j \circ f \circ \phi_j^{-1})^* \omega_i = (\psi_i \circ f \circ \phi_j^{-1} \circ \phi_{ij})^* \omega_i.$$

By the definitions of ϕ_j, ϕ_{ij}, and h_i, we have

$$\phi_{ij}^* \chi_j = (\psi_i \circ f \circ \phi_i^{-1})^* \omega_i = h_i^* \omega_i = \chi_i.$$

This establishes what we want on $\phi_i(U_i - P_i)$. At each point p in $\phi(P_i)$ we must have $(g_i \circ g_i)h_i' = (g_j \circ g_j)h_j'$, because the value at p is determined by the neighboring values, which are the same.

We say that a pullback $f^* \omega$ has **corresponding centers** if, for every chart C_i in R_1 that is centered at p, the corresponding chart D_i in R_2 is centered at $f(p)$.

Fix a pullback $f^* \omega$ with corresponding centers. The following proposition relates the order of $f^* \omega$ at a point p with the order of ω at $f(p)$ and the multiplicity of f at p:

Proposition: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f\colon R_1 \to R_2$ be a holomorphic map with the chart inclusion property. Let $\omega = \{g_i \, dz\}$ be a meromorphic one form on R_2, and let $f^* \omega$ be a pullback with corresponding centers. Then for any point p in T, we have

$$\operatorname{ord}_p \, (f^* \omega) = (1 + \operatorname{ord}_{f(p)} \, \omega) \operatorname{mult}_p \, f - 1.$$

To prove the proposition, we need a lemma:

Lemma: Let $U \subseteq \mathbf{C}$ be an open set containing zero, let f be a meromorphic function on U, and let g be a holomorphic function on U. Then

$$\operatorname{ord}_0 \, (f \circ g) = (\operatorname{ord}_0 \, f)(\operatorname{ord}_0 \, g).$$

Proof: Let $m = \operatorname{ord}_0 f$, and let $n = \operatorname{ord}_0 g$. We must prove that $\operatorname{ord}_a(f \circ g) = mn$. Consider the Laurent series expansions of f and g at 0. We have $f(z) = z^m P(z)$ and $g(z) = z^n Q(z)$, where P and Q are power series with nonzero constant terms, and $n \geq 0$. Then

$$(f \circ g)(z) = z^{mn} [Q^m(z)][(P \circ z^n Q)(z)],$$

where Q^m and $P \circ z^n Q$ are power series with nonzero constant terms. Therefore

$$(f \circ g)(z) = z^{mn} H(z),$$

where $H = Q^m (P \circ z^n Q)$ is a power series with a nonzero constant term, and so $\operatorname{ord}_0 \, (f \circ g) = mn$, as required. □

Proof of the proposition: Fix a chart $C_i = (U_i, \phi_i)$ centered at p. Then by assumption the corresponding chart $D_i = (V_i, \psi_i)$ is centered at $f(p)$. Let h_i be the holomorphic function $\psi_i \circ f \circ \phi_i^{-1}$. Then by definition we have

$$\operatorname{ord}_p \, (f^* \omega) = \operatorname{ord}_0 \, (f^* \omega)_i = \operatorname{ord}_0 \, (g_i \circ h_i) h_i'.$$

Let the Laurent series expansions for $g_i \circ h_i$ and for h_i' be $z^m P$ and $z^n Q$, where P and Q are power series with nonzero constant terms. Then $(g_i \circ h_i) h_i' = z^{m+n} PQ$, so

$$\operatorname{ord}_p \, (f^* \omega) = \operatorname{ord}_0 \, (g_i \circ h_i) + \operatorname{ord}_0 \, h_i',$$

and by the lemma we have

$$\operatorname{ord}_p \, (f^* \omega) = (\operatorname{ord}_0 \, g_i)(\operatorname{ord}_0 \, h_i) + \operatorname{ord}_0 \, h_i'. \tag{4}$$

Now consider the following facts:

1. By definition, $\operatorname{ord}_0 \, g_i = \operatorname{ord}_{f(p)} \, \omega$.
2. $h_i(0) = 0$, so the power series expansion for h_i has a zero constant term. Therefore $\operatorname{ord}_0 \, h_i = \operatorname{mult}_p \, f$ and $\operatorname{ord}_0 \, h_i' = \operatorname{mult}_p \, f - 1$. See *Holomorphic Maps Between Riemann Surfaces*, § 1.3.

Putting these facts together with (4) and collecting terms yields the result. □

1.4. The Residue Theorem for Compact Riemann Surfaces

Let $R = (T, A)$ be a Riemann surface, and let $\omega = \{\omega_i\}$ be a meromorphic one form on R. Fix a point p in T. We define the **residue** of ω at p, written $\operatorname{Res}_p \, \omega$, to be the residue at $\phi_i(p)$ of f_i, where $C_i = (U_i, \phi_i)$ is any chart of A

containing p, and $\omega_i = f_i\ dz$. From *Holomorphic Maps Between Riemann Surfaces*, § 1.5, we know that the residue is independent of the choice of chart, so this definition is valid.

The following theorem, called the **residue theorem**, is a basic result in the theory of integration on compact Riemann surfaces:

> *Theorem (Residue Theorem): Let $R = (T, A)$ be a compact Riemann surface, and let $\omega = \{\omega_i\}$ be a meromorphic one form on R. Then*
>
> $$\sum_{p \in T} \mathrm{Res}_p\ \omega = 0.$$

Note that the sum in the statement of the theorem is finite, because (1) the residue at p is nonzero only if the order of ω at p is negative; and (2) the set of points of ω of negative order is a closed and discrete subset of the compact space T, and therefore finite.

To prove the theorem, we need some definitions and lemmas.

Up to an invertible differentiable map, we know that T is a sphere with $g \geq 0$ handles, and that T has a triangulation. See *Holomorphic Maps Between Riemann Surfaces*, § 2.1 and § 2.2. We call a triangulation τ **oriented** if each triangle t_i in τ has an orientation, i.e., a direction of traversing each edge of t_i that stays consistent when passing from one edge to another. Each triangle in τ has exactly two possible orientations. We shall say that an oriented triangulation τ **sums to zero** if, for each edge e in τ, the adjacent triangles sharing e traverse e in opposite directions.

Consider a triangle t_i in τ. We may consider each oriented edge e_{ij} of t_i as a path σ_{ij} in R. See *Complex Charts on Topological Surfaces*, § 2.3 and § 4.3. We define the **boundary chain** ∂t_i of a triangle t_i to be the sum of the oriented edge paths. We define the **triangluation chain** γ to be the sum of the boundary chains ∂t_i in τ. Then τ sums to zero if and only if $\gamma = 0$, because the opposite pairs of edges cancel out.

> *Lemma 1: Let $R = (T, A)$ be a compact Riemann surface. Then for any integer $n > 0$, T has an oriented triangulation with at least n triangles that sums to zero.*

Proof: We proceed by induction on the genus g of R. In the case $g = 0$, we can triangulate a sphere with a tetrahedron. It is easy to orient the edges of this triangulation so that it sums to zero. Given any oriented triangulation τ with $m < n$ triangles, we can add more triangles as follows. Pick a triangle t_i in τ, pick a point p in the interior of t_i, and add edges from p to each of the vertices of t_i. Then we can orient the new edges so that the resulting triangulation sums to zero. We can repeat this process until τ has at least n triangles. This proves the result for the case $g = 0$.

Now assume the result for a compact Riemann surface R_{g-1} with $g - 1$ handles. We can form a compact Riemann surface R_g of genus g by attaching a triangulated sphere S to R_{g-1} at each of two triangles. We can connect each pair of triangles with a triangular cylinder, and we can place the lines of the cylinder, twisting if necessary so that the triangles have matching orientations. Then we can triangulate the cylinder by adding a diagonal edge to each of the three rectangular faces, and we can orient the new edges so that the resulting triangulation sums to zero. □

> *Lemma 2: Let $U \subseteq \mathbf{C}$ be an open set, and let ω be a meromorphic one form on U. Let t be an oriented triangle contained in U. Then*
>
> $$\int_{\partial t} \omega = 2\pi i \sum_{p \in t} \mathrm{Res}_p\ \omega.$$

Proof: Construct a closed path σ in U by smoothing the corners of ∂t. Then by *Calculus Over the Complex Numbers*, § 6.1, we have

$$\int_{\sigma} \omega = 2\pi i \sum_{p \in t} \mathrm{Res}_p\ \omega.$$

Note that all the terms of the sum are zero, except at the finite set of points p that are poles of ω. By moving the smoothed corners of σ close enough to the corners of ∂t, we can make the integral over σ arbitrarily close to the integral over ∂t. But the integral over σ is constant. This proves the result. □

Proof of the theorem: By Lemma 1, we can choose an oriented triangulation τ of T that sums to zero, and by adding enough triangles, we can ensure that each triangle lies inside a chart of A. Further, because the set of poles of ω is finite, we can ensure that no pole lies on a vertex of τ. Let γ be the sum of the boundary chains ∂t_i. Then $\gamma = 0$, so we have

$$\int_\gamma \omega = 0. \tag{5}$$

On the other hand, by the definition of integration on a Riemann surface (see *Complex Charts on Topological Surfaces*, § 2.3 and § 4.3), the integral in (5) is the sum of the integrals over the oriented triangles. Because each triangle is contained in a chart domain, each such integral is a complex integral in an open subset of $\mathbf{C}$. Therefore by Lemma 2, we have

$$\int_\gamma \omega = 2\pi i \sum_{p \in P} \mathrm{Res}_p\ \omega. \tag{6}$$

Together, (5) and (6) establish the result. □

Example 1: Let $R = \mathbf{C}_\infty$, and let ω be the meromorphic one form defined in § 1.1. Then ω has a pole of order three at ∞ and no other poles. So the residue of ω at every point is zero.

Example 2: Again let $R = \mathbf{C}_\infty$. Define $\omega_1 = 1/z\ dz$ and $\omega_2 = -(1/z)\ dz$. As shown in § 1.1, this definition gives a valid meromorphic one form on R. The residue of ω is 1 at zero, -1 at ∞, and zero everywhere else. The sum of the residues is zero, as expected.

Example 3: Let $R = (T, A)$ be a compact Riemann surface, and let f be a meromorphic function on R. Let ω be the meromorphic one form on R given by $\omega_i = df_i/f_i$, where f_i is the local function $f \circ \phi_i^{-1}$ associated with the chart $C_i = (U_i, \phi_i)$ of A. From *Holomorphic Maps Between Riemann Surfaces*, § 1.5, we know that (1) ω is a meromorphic one form on R and (2) for each point p in T, we have $\mathrm{Res}_p\ \omega = \mathrm{ord}_p\ f$. By the residue theorem, the sum of the residues of ω over the points of T is zero. Therefore the sum of the orders of f is zero. Thus we have proved, in a different way, the result shown in § 3.6 of *Holomorphic Maps Between Riemann Surfaces*, namely that the sum of the orders of a meromorphic function over the points on a compact Riemann surface is zero.

2. Spaces of Functions and One Forms

In this section we define several spaces of functions and of one forms that are important in the study of Riemann surfaces. Throughout this section, $R = (T, A)$ denotes a Riemann surface.

2.1. Meromorphic Functions and One Forms

The space $M(R)$: We write $M(R)$ to denote the set of meromorphic functions on R. $M(R)$ is a field, according to the following rules:

1. $f + g = p \mapsto f(p) + g(p)$
2. $-f = p \mapsto -f(p)$
3. $fg = p \mapsto f(p)g(p)$
4. $1/f = p \mapsto 1/f(p)$

$M(R)$ is also a vector space over $\mathbf{C}$, with scalar multiplication given by the rule $af = p \mapsto af(p)$.

The space $M^{(1)}(R)$: We write $M^{(1)}(R)$ to denote the set of meromorphic one forms on R. $M^{(1)}(R)$ is a vector space over $\mathbf{C}$, with addition given by the rule

$$\{f_i\ dz\} + \{g_i\ dz\} = \{(f_i + g_i)\ dz\}$$

and scalar multiplication given by the rule

$$a \cdot \{f_i\ dz\} = \{af_i\ dz\}.$$

The addition rule is valid because we have

$$\begin{aligned} \phi_{ij}^*[(f_j + g_j)\ dz] &= [(f_j + g_j) \circ \phi_{ij}]\phi'_{ij}\ dz \\ &= (f_j \circ \phi_{ij})\phi'_{ij}\ dz + (g_j \circ \phi_{ij})\phi'_{ij}\ dz \\ &= \phi_{ij}^*(f_j\ dz) + \phi_{ij}^*(g_j\ dz) \end{aligned}$$

$$= f_i \; dz + g_i \; dz = (f_i + g_i) \; dz.$$

A similar computation shows that $\phi_{ij}^*(af_j) = af_i$.

$M^{(1)}(R)$ is also a vector space over the field $M(R)$, with scalar multiplication given by the rule

$$f \cdot \{g_i \; dz\} = \{f_i g_i \; dz\},$$

where f is a meromorphic function on R, and $f_i = f \circ \phi_i^{-1}$. A similar computation to the one given above shows that $\phi_{ij}^*(f_j g_j) = f_i g_i$.

2.2. Holomorphic Functions and One Forms

The space $O(R)$: We write $O(R)$ to denote the set of holomorphic functions on R. $O(R)$ is a ring and is a subring of $M(R)$. It is also a vector space over $\mathbf{C}$ and is a subspace of $M(R)$.

The space $O^{(1)}(R)$: We write $O^{(1)}(R)$ to denote the set of holomorphic one forms on R. $O^{(1)}(R)$ is a subspace of $M^{(1)}(R)$ as a vector space over $\mathbf{C}$ and as a vector space over $M(R)$.

3. Divisors

We have seen several examples of functions that associate, to each point p on a Riemann surface R, information about the local behavior at p of a function, map, or differential form on R. Examples include the order function $p \mapsto \operatorname{ord}_p f$ for a meromorphic function f, the multiplicity function $p \mapsto \operatorname{mult}_p f$ for a holomorphic map f, and the residue function $p \mapsto \operatorname{Res}_p \omega$ for a meromorphic one form ω. We now develop some standard notation for and properties of this kind of function.

3.1. The Definition of a Divisor

Let $R = (T, A)$ be a Riemann surface. A **divisor** on R is a function $D: T \to G$, where G is an additive group. This group is often, but not always, the integers. We write a divisor as a formal sum, as follows:

$$D = \sum_{p \in T} g_p \; p, \tag{1}$$

where $g_p = D(p)$. By associating a value g_p to each point p, a divisor gives a very general way to record local information at each point on a Riemann surface.

As usual, we write $D_1 + D_2$ to denote the function $p \mapsto D_1(p) + D_2(p)$, and we write $-D$ to denote the function $p \mapsto -D(p)$. Here $+$ denotes addition in G, and $-$ denotes the additive inverse in G. We also write $D_1 - D_2$ as a shorthand for $D_1 + (-D_2)$, in the usual way. Finally, we write 0 to denote the divisor $p \mapsto 0$, where the second 0 means the identity in the additive group G. These definitions make the set of all divisors $D: T \to G$ for a fixed additive group G into an additive group.

Fix a divisor D. The **support** of D is the set of points p in T for which $D(p) \neq 0$. When the support of D is a finite set, we say that D has **finite support**. In this case, the sum (1) is a finite sum.

3.2. Principal Divisors

Let $R = (T, A)$ be a Riemann surface. A **principal divisor** on R, written (f), is the divisor $D: R \to \mathbf{Z}$ corresponding to the order function for a meromorphic function f on R that is not identically zero:

$$(f) = \sum_{p \in T} (\operatorname{ord}_p f) \; p.$$

In other words, (f) is the function $p \mapsto \operatorname{ord}_p f$. Note that (f) is not defined for the function f that is identically zero, because the order of this function at every point is ∞, which is not an integer.

Let (f) and (g) be principal divisors on R, and let p be a point in T. For any chart C_i containing p, f_i has a Laurent series expansion $z^m P$ at p_i and g_i has a Laurent series expansion $z^n Q$ at p_i, where P and Q are power series with nonzero constant terms.

Divisors of products: $f_i g_i$ has a Laurent series expansion $z^{m+n} PQ$ at p_i. Since PQ has a nonzero constant term, the order of $f_i g_i$ at p_i is $m + n$. Therefore we have $(f + g)(p) = (f)(p) + (g)(p)$ for every p, i.e.,

$$(fg) = (f) + (g). \tag{2}$$

Divisors of inverses: $1/f_i$ has a Laurent series expansion $z^{-m}H$, where $H = 1/P$ is a power series with a nonzero constant term. See *Calculus Over the Complex Numbers*, § 4.1 and § 4.2. Since H has a nonzero constant term, the order of $1/f_i$ at p_i is $-m$. Therefore we have $(1/f)(p) = -(f)(p)$ for every p, i.e.,

$$(1/f) = -(f) \tag{3}$$

Divisors of ratios: From (2) and (3) and the definition of $D_1 - D_2$ we immediately obtain

$$(f/g) = (f) - (g) \tag{4}$$

3.3. Canonical Divisors

Let $R = (T, A)$ be a Riemann surface. A **canonical divisor** on R, written (ω), is the divisor corresponding to the order function for a meromorphic one form ω on R that is not identically zero:

$$(\omega) = \sum_{p \in T} (\mathrm{ord}_p\ \omega)\ p.$$

In other words, (ω) is the function $p \mapsto \mathrm{ord}_p\ \omega$. For the same reason stated in § 3.2, (ω) is not defined for the meromorphic one form ω that is identically zero.

Let $\omega = \{g_i\ dz\}$ be a meromorphic one form on R, and let f be a meromorphic function on R. By the definition of the vector space $M^{(1)}(R)$ over the field $M(R)$ in § 2.1, we can multiply f by ω, yielding the meromorphic one form $f\omega$. By this definition, the order of $f\omega$ at a point p is the order of $f_i g_i$ at p. Further, the order of ω at p is the order of g_i at p. Therefore by the argument made in § 3.2, we obtain the formula

$$(f\omega) = (f) + (\omega). \tag{5}$$

Principal and canonical divisors are further related in the following way:

Proposition: Let $R = (T, A)$ be a Riemann surface. Let ω and χ be meromorphic one forms on R, with ω not identically zero. Then there exists a unique meromorphic function f on R such that $\chi = f\omega$.

Proof: Let $\omega = \{g_i\ dz\}$ and $\chi = \{h_i\ dz\}$. For each chart $C_i = (U_i, \phi_i)$, let f_i be the meromorphic function h_i/g_i on $\phi_i(U_i)$. We wish to show that $f = \{f_i\}$ is a meromorphic function on R. For any pair of charts C_i and C_j, we have

$$\phi_{ij}^*\ f_j = \frac{h_j \circ \phi_{ij}}{g_j \circ \phi_{ij}}.$$

Because $\phi_{ij}'(z) \neq 0$ on its domain of definition, we can write

$$\phi_{ij}^*\ f_j = \frac{(h_j \circ \phi_{ij})\phi_{ij}'}{(g_j \circ \phi_{ij})\phi_{ij}'}.$$

Then by the definitions of ω, χ, and f_i, we have

$$\phi_{ij}^*\ f_j = \frac{h_i}{g_i} = f_i.$$

Thus f is a meromorphic function on R that satisfies the statement of the proposition. From the construction, it is clear that f is unique. □

Corollary: Let $R = (T, A)$ be a Riemann surface. Let ω be a meromorphic one form on R, and let g be a nonconstant meromorphic function on R. Then there exists a unique meromorphic function f on R such that $\omega = f\ dg = \{f_i\ dg_i\}$.

Proof: From § 4.3 of *Complex Charts on Topological Surfaces*, we know that if g is holomorphic, then $dg = \{dg_i\} = \{g_i'\ dz\}$ is a one form on R. The same proof goes through when g is meromorphic. Because g is nonconstant, $dg \neq 0$. The result then follows from the proposition. □

3.4. Linear Equivalence of Divisors

Let R be a Riemann surface, and let D_1 and D_2 be divisors on R. We say that D_1 and D_2 are **linearly equivalent** and write $D_1 \sim D_2$ if $D_1 - D_2$ is a principal divisor, i.e., if there exists a meromorphic function f on R such that $D_1 - D_2 = (f)$.

Proposition: Let $R = (T, A)$ be a Riemann surface, and let (ω_1) and (ω_2) be canonical divisors on R. Then $(\omega_1) \sim (\omega_2)$.

Proof: The result follows from formula (5) and the proposition stated in § 3.3, together with the observation that neither ω_1 nor ω_2 can be identically zero. □

3.5. The Degree of a Divisor on a Compact Riemann Surface

Let $R = (T, A)$ be a Riemann surface, and let $D: R \to \mathbf{Z}$ be an integer-valued divisor with finite support. We define the **degree** of D, written deg D, as follows:

$$\deg D = \sum_{p \in T} D(p). \tag{6}$$

Note that because D has finite support, (6) is a finite sum.

Principal divisors: Let R be a compact Riemann surface, and let f be a meromorphic function on R. Then the set of points p where f has nonzero order is finite, so the degree of the principal divisor (f) is well-defined. Further, we have

$$\boxed{\deg (f) = 0.}$$

This statement is exactly the result proved in § 3.6 of *Holomorphic Maps Between Riemann Surfaces* and again in example 3 of § 1.4 of this document.

Canonical divisors: On a compact Riemann surface, the degree of a canonical divisor (ω) is also well-defined. We now prove several results about the degree of a canonical divisor on a compact Riemann surface.

Proposition: Let R be a compact Riemann surface, and let (ω_1) and (ω_2) be canonical divisors on R. Then $\deg (\omega_1) = \deg (\omega_2)$.

Proof: By 3.4, there exists a principal divisor (f) such that $(\omega_2) = (\omega_1) + (f)$. By the previous result, $\deg (f) = 0$. Since the degree function is linear, the result follows. □

Theorem: Let R be a compact Riemann surface of genus g. If R has a nonconstant meromorphic function, then R has a canonical divisor with degree $2g - 2$.

Proof: Let $R = (T, A)$, let g be the meromorphic function, and let $f: R \to \mathbf{C}_\infty$ be the associated holomorphic map to the Riemann sphere. We will use f to construct a canonical divisor on R with degree $2g - 2$.

Let ω be the meromorphic one form on $\mathbf{C}_\infty$ given by $\omega_1 = dz$ and $\omega_2 = -(1/z^2)\, dz$. As discussed in § 1.3, f has the chart inclusion property; and by translating we can choose corresponding charts with corresponding centers. Therefore we may construct a pullback $f^*\omega$ with corresponding centers.

Now consider the degree of the canonical divisor $(f^*\omega)$. By the proposition in § 1.3, we have

$$\deg (f^*\omega) = \sum_{p \in T} \operatorname{ord}_p f^*\omega = \sum_{p \in T} [(1 + \operatorname{ord}_{f(p)} \omega) \operatorname{mult}_p f - 1].$$

The order of ω is -2 at ∞ and zero everywhere else. Therefore we have

$$\deg (f^*\omega) = \sum_{p \in T - f^{-1}(\infty)} (\operatorname{mult}_p f - 1) + \sum_{p \in f^{-1}(\infty)} (-\operatorname{mult}_p f - 1).$$

Rearranging terms, we have

$$\deg (f^*\omega) = \sum_{p \in T} (\operatorname{mult}_p f - 1) + \sum_{p \in f^{-1}(\infty)} -2 \operatorname{mult}_p f. \tag{7}$$

By the Hurwitz formula (*Holomorphic Maps Between Riemann Surfaces*, § 3.7) with $g_2 = 0$, the first term of (7) is $2g - 2 + 2 \deg f$. By the definition of the degree of a holomorphic map (*Holomorphic Maps Between Riemann Surfaces*, § 3.5), the second term of (7) is $-2 \deg f$. Adding these terms yields the result. □

Corollary 1: Let R be a compact Riemann surface of genus g. If R has a nonconstant meromorphic function, then all canonical divisors on R have degree $2g - 2$.

Proof: This statement follows from the theorem and from the proposition. □

Corollary 2: Let R be a compact Riemann surface of genus g. If R has at least two distinct nonconstant meromorphic one forms, then all canonical divisors on R have degree $2g - 2$.

Proof: This statement follows from the proposition stated in § 3.3 and from Corollary 1. □

3.6. The Spaces $L(D)$ and $L^{(1)}(D)$

The partial ordering on divisors: Let $\mathbf{Z}^R$ be the set of all integer-valued divisors on R, i.e., the set of all divisors $D: T \to \mathbf{Z}$. As noted in § 3.1, $\mathbf{Z}^R$ is an additive group. We make S into a partially ordered set as follows. If D_1 and D_2 are two elements of $\mathbf{Z}^R$, then we say that $D_1 \geq D_2$ if and only if $D_1(p) \geq D_2(p)$ for all p in T. From the definition of the element zero of $\mathbf{Z}^R$, it follows that $D \geq 0$ if and only if $D(p) \geq 0$ for all p in T. As usual, we say $D_1 \leq D_2$ if and only if $D_2 \geq D_1$.

The space $L(D)$: Let D be an integer-valued divisor on R. We define $L(D)$ to be the set of meromorphic functions f on R such that $(f) \geq -D$.

One may ask why the definition of $L(D)$ uses $-D$ instead of D. The motivation seems to be that we are primarily interested in bounding poles.

Fix an integer-valued divisor D, a meromorphic function f in $L(D)$, and a point p in T. Let $n = D(p)$. Then we have the following:

1. If $n > 0$, then f may or may not have a pole at p, and if it does, then the pole is of order no greater than n.
2. If $n = 0$, then f does not have a pole at p. It may or may not have a zero at p.
3. If $n < 0$, then f has a zero at p of at least order n.

For any integer-valued divisor D on R, $L(D)$ is a subset of $M(R)$, and it is closed under addition and under multiplication by a complex number. Therefore $L(D)$ is a vector space over $\mathbf{C}$ and is a subspace of $M(R)$.

The space $L(0)$ consists of exactly the meromorphic functions on R with no poles, i.e., the holomorphic functions on R. Therefore we have $L(0) = O(R)$. When R is compact, we have that $O(R)$ consists of the constant functions on R. See *Holomorphic Maps Between Riemann Surfaces*, § 3.2. In this case, we have $L(0) = \mathbf{C}$.

Let D_1 and D_2 be integer-valued divisors on R such that $D_1 \sim D_2$. Then $L(D_1)$ and $L(D_2)$ are isomorphic as vector spaces. Indeed, we have $D_1 - D_2 = (g)$ for some meromorphic function g, and for any f in $L(D_1)$, by equation (2) we have

$$(gf) = (g) + (f) \geq (g) - D_1 = -D_2,$$

so (gf) is an element of $L(D_2)$. Therefore multiplication by g is a linear map from $L(D_1)$ to $L(D_2)$. Since $D_2 - D_1 = -(g) = (1/g)$, by the same argument multiplication by $1/g$ is a linear map from $L(D_2)$ to $L(D_1)$. The composition of these two linear maps the identity map, so each map is an isomorphism.

The space $L^{(1)}(D)$: Let D be an integer-valued divisor on R. We define $L^{(1)}(D)$ to be the set of meromorphic one forms ω on R such that $(\omega) \geq -D$. $L^{(1)}(D)$ is a vector space over $\mathbf{C}$ and is a subspace of $M^{(1)}(R)$.

The space $L^{(1)}(0)$ consists of exactly the meromorphic one forms on R with no poles, i.e., the holomorphic one forms on R. Therefore we have $L^{(1)}(0) = O^{(1)}(R)$.

Let D_1 and D_2 be integer-valued divisors on R such that $D_1 \sim D_2$. Then $L^{(1)}(D_1)$ and $L^{(1)}(D_2)$ are isomorphic as vector spaces. The argument given above for $L(D)$ goes through, except that we use equation (5) instead of equation (2).

4. Laurent Polynomials

To state and prove the Riemann-Roch theorem, we will need to consider finite prefixes of Laurent series. Such a prefix is a finite series

$$\sum_{j=m}^{n} a_j z^j, \tag{1}$$

where m and n are integers. We will call such a finite series a **Laurent polynomial**. Equivalently, a Laurent polynomial is a Laurent series in which the coefficients a_j are zero for all j greater than some integer n.

Fix a Laurent polynomial P. We say that P is **bounded by** the integer n if the coefficients of P are all zero at indices n and greater.

4.1. Laurent Series and Polynomial Divisors

First we consider mappings that associate Laurent series and Laurent polynomials to the points of a Riemann surface.

Laurent series divisors: Let $\mathbf{L}$ denote the set of all Laurent series. It is a complex vector space, i.e., a vector space over $\mathbf{C}$.

Let $R = (T, A)$ be a Riemann surface, and fix a mapping $\Lambda\colon T \to \mathbf{L}$. We call Λ a **Laurent series divisor**. To each point p in T, it assigns a Laurent series $\Lambda(p)$. The set of all Laurent series divisors on R is a complex vector space, which we denote $\mathbf{L}^R$.

We may embed $M(R)$, the space of meromorphic functions on R, in $\mathbf{L}^R$ as follows. For each point p in T, choose a chart C_p of A centered at p. Then for each meromorphic function f on R, f_p has a Laurent series expansion l_p at p. The mapping $\Lambda_f = p \mapsto l_p$ is a Laurent series divisor on R, and $f \mapsto \Lambda_f$ is an injective map from $M(R)$ to $\mathbf{L}^R$.

The mapping $f \mapsto \Lambda_f$ is not surjective, because in general we do not get a meromorphic function on R by assigning arbitrary Laurent series to points of R. For example, the divisor $\Lambda(0) = 0$, $\Lambda(p \neq 0) = 1$ on the Riemann sphere does not correspond to any meromorphic function.

Laurent polynomial divisors: Let $\mathbf{P}$ denote the set of all Laurent polynomials. It is a complex vector space and a subspace of $\mathbf{L}$.

Let $R = (T, A)$ be a Riemann surface, and fix a mapping $\Pi\colon R \to \mathbf{P}$. We call Π a **Laurent polynomial divisor**. To each point p in T, it assigns a Laurent polynomial $\Pi(p)$.

Let Π be a Laurent polynomial divisor on R, and let D be an integer-valued divisor on R. We say that Π is **bounded by** D if for each p in T, $\Pi(p)$ is bounded by $D(p)$.

The set of all Laurent polynomial divisors on R is a complex vector space, which we denote $\mathbf{P}^R$. $\mathbf{P}^R$ is a subspace of $\mathbf{L}^R$. The set of all Laurent polynomial divisors on R with finite support is a subspace of $\mathbf{P}^R$, which we denote $\mathbf{P}_0^R$.

4.2. Truncation Maps

Next we consider mappings that zero out the coefficients of a Laurent series after a certain point, converting them to Laurent polynomials. We call these maps **truncation maps**.

Truncation by an integer: Let n be an integer. We define the truncation map $t_n\colon \mathbf{L} \to \mathbf{P}$ as follows:

$$\sum_{j=m}^{\infty} a_j z^j \mapsto \sum_{j=m}^{n-1} a_j z^j.$$

That is, for any Laurent series l, $t_n(l)$ is the Laurent polynomial consisting of l with the terms of order n and higher zeroed out.

Truncation by a divisor: Let $R = (T, A)$ be a Riemann surface, and let D be an integer-valued divisor on R. We define the truncation map $t_D\colon \mathbf{L}^R \to \mathbf{P}^R$ as follows:

$$t_D(\Lambda) = (p \mapsto t_{-D(p)}(\Lambda(p)).$$

That is, for any Laurent series divisor Λ, $t_D(\Lambda)$ is the Laurent polynomial divisor that maps p to the Laurent series $\Lambda(p)$ with the terms of order $-D(p)$ and higher zeroed out.

The truncation map t_D is a linear map, so if V is any subspace of $\mathbf{L}^R$, then $t_D(V)$ is a complex vector space. In particular, $t_D(\mathbf{P}_0^R)$ is a complex vector space. It contains exactly the Laurent polynomial divisors on R that have finite support and that are bounded by $-D$.

4.3. The Vector Space $H^1(D)$

Let R be a compact Riemann surface, and let D be an integer-valued divisor on R with finite support. As observed in § 4.1, we may treat $M(R)$ as a subspace of $\mathbf{L}^R$. Therefore, we may apply the truncation map t_D to $M(R)$ to obtain the vector space $t_D(M(R))$. Further, if Λ_f is the Laurent series divisor corresponding to a meromorphic function f on R, then $t_D(\Lambda_f)$ has finite support. This is because (1) at all but a finite number of points p we have $D(p) = 0$, so

$$t_D(\Lambda_f)(p) = t_0(\Lambda_f(p));$$

and (2) at all but a finite number of points p we have that $\Lambda_f(p)$ has no negative terms, so $t_0(\Lambda_f(p)) = 0$. Therefore, we have

$$t_D(M(R)) \subseteq t_D(\mathbf{P}_0^R),$$

and we may construct the quotient space $t_D(\mathbf{P}_0^R)/t_D(M(R))$. This is the space of Laurent polynomials of $t_D(\mathbf{P}_0^R)$ subject to the relation that two polynomials are equivalent in the quotient space if they differ by an element of $t_D(M(R))$. We call this quotient space $H^1(D)$, i.e., we define

$$H^1(D) = t_D(\mathbf{P}_0^R)/t_D(M(R)).$$

$H^1(D)$ measures the amount by which $t_D(M(R))$ fails to equal $t_D(\mathbf{P}_0^R)$.

The name $H^1(D)$ comes from the concept of **cohomology**, which you can read about in my paper *Definitions for Commutative Algebra*. Cohomology is a general way to study sequences of maps in which the image of one map lies in the kernel of the next map. Such sequences are called **cochain complexes**.

In terms of cohomology, for a fixed Riemann surface R and a fixed integer-valued divisor D on R, we can write the following sequence of maps:

$$0 \to L(D) \to M(R) \xrightarrow{t_D} t_D(\mathbf{P}_0^R) \to 0. \tag{2}$$

Here 0 means the trivial vector space consisting of just the element zero. The first two arrows are the inclusion maps, and the last arrow is the map taking every element to zero. In the sequence (2), the image of each arrow lies in the kernel of the next arrow. In particular, $L(D)$ is exactly the kernel in $M(R)$ of t_D, because a meromorphic function f has order at least $-D(p)$ at a point p if and only if its Laurent series at p has all zero coefficients below order $-D(p)$, i.e., the truncation of its Laurent series at p under $t_{-D(p)}$ is zero. Therefore, the sequence (2) is a cochain complex. The space $H^1(D)$ is the **cohomology space** associated with $t_D(\mathbf{P}_0^R)$, i.e., the kernel $t_D(\mathbf{P}_0^R)$ of the map $t_D(\mathbf{P}_0^R) \to 0$ modulo the image $t_D(M(R))$ of the map $M(R) \xrightarrow{t_D} t_D(\mathbf{P}_0^R)$.

4.4. The Dimensions of $L(D)$ and $H^1(D)$

Let D be an integer-valued divisor with finite support on a compact Riemann surface. We now assert two important facts about the spaces $L(D)$ and $H^1(D)$. First, we assert a result about the dimension of $H^1(D)$:

Proposition 1: Let R be a compact Riemann surface, and let D be an integer-valued divisor on R with finite support. Then $H^1(D)$ is a finite-dimensional vector space over $\mathbf{C}$.

For the proof, see [Miranda 1995], VI, Proposition 2.7.

Next we assert a result about the the dimension of $L(D)$:

Proposition 2: Let R be a compact Riemann surface, and let D be an integer-valued divisor on R with finite support. Then $L(D)$ is a finite-dimensional vector space over $\mathbf{C}$*, and we have*

$$\dim L(D) - \deg D = \dim H^1(D) - \dim H^1(0) + 1.$$

To prove Proposition 2, we need a lemma.

Let D_1 and D_2 be integer-valued divisors on $R = (T, A)$ with finite support, and suppose $D_1 \leq D_2$. Let p_1 and p_2 be Laurent polynomials in $t_{D_1}(\mathbf{P}_0^R)$ that are equivalent in $H^1(D_1)$. This means that $p_1 - p_2$ lies in $t_{D_1}(M(R))$. Because $D_1 \leq D_2$, at each point q of T, t_{D_2} zeros out at least as many coefficients as t_{D_1}, so $t_{D_2}(p_1 - p_2) = t_{D_2}(p_1) - t_{D_2}(p_2)$ is an element of $t_{D_2}(M(R))$. Therefore $t_{D_2}(p_1)$ is equivalent to $t_{D_2}(p_2)$ in $H^1(D_2)$, and t_D induces a well-defined map

$$h_{D_1,D_2}\colon H^1(D_1) \to H^1(D_2).$$

We now assert a lemma about the kernel of this map h_{D_1,D_2}:

Lemma: Let R be a compact Riemann surface. Let D_1 and D_2 be integer-valued divisors on R with finite support, and suppose $D_1 \le D_2$. Then the kernel of the map h_{D_1,D_2} is finite-dimensional over $\mathbf{C}$, and we have

$$\dim\,(\ker\, h_{D_1,D_2}) = (\dim\, L(D_1) - \deg\, D_1) - (\dim\, L(D_2) - \deg\, D_2).$$

For the proof, see [Miranda 1995], VI, Lemma 2.3.

Proof of Proposition 2: Since $H^1(D_1)$ and $H^1(D_2)$ are finite-dimensional (Proposition 1), by elementary linear algebra we have

$$\dim\,(\ker\, h_{D_1,D_2}) = \dim\, H^1(D_1) - \dim\, H^1(D_2).$$

As observed in § 3.6, $L(0) = \mathbf{C}$, so $\dim\, L(0) = 1$. If $0 \le D$, then by the lemma with $D_1 = 0$ and $D_2 = D$ we have

$$\begin{aligned} \dim\, L(D) - \deg\, D &= -\dim\,(\ker\, h_{0,D}) + (\dim\, L(0) - \deg\, 0)) \\ &= -(\dim\, H^1(0) - \dim\, H^1(D)) + (1 - 0) \\ &= \dim\, H^1(D) - \dim\, H^1(0) + 1. \end{aligned}$$

Otherwise by the lemma with $D_1 = D$ and $D_2 = 0$ we have

$$\begin{aligned} \dim\, L(D) - \deg\, D &= \dim\,(\ker\, h_{D,0}) + (\dim\, L(0) - \deg\, 0)) \\ &= \dim\, H^1(D) - \dim\, H^1(0) + (1 - 0) \\ &= \dim\, H^1(D) - \dim\, H^1(0) + 1. \end{aligned}$$

□

5. Serre Duality

In this section, we assert a fact called **Serre duality** that is key to the proof of the Riemann-Roch theorem. Let R be a compact Riemann surface, and let D be an integer-valued divisor on R with finite support. We denote by $H^1(D)^*$ the complex vector space of linear maps $\lambda\colon H^1(D) \to \mathbf{C}$. This space is called the **dual space** of $H^1(D)$. Recall from § 3.6 that $L^{(1)}(-D)$ is the vector space of meromorphic one forms ω on R such that $(\omega) \ge D$. Serre duality asserts the existence of an isomorphism between $L^{(1)}(-D)$ and $H^1(D)^*$.

5.1. The Residue Map

To formulate Serre duality, we need to define a map

$$\mathrm{Res}\colon L^{(1)}(-D) \to H^1(D)^* \tag{1}$$

called the **residue map**. To do this, we will need a lemma. In stating this lemma, we will write f_a to denote the Laurent series expansion at a point a of a meromorphic complex function f.

Lemma: Let U be an open subset of $\mathbf{C}$. Fix a point a in U and an integer n. Let f be a meromorphic function on U, and let ω be a meromorphic one form on U such that $\mathrm{ord}_a\, \omega \ge n$. *Then*

$$\mathrm{Res}_a\, f\omega = \mathrm{Res}_a\, t_{-n}(f_a)\omega.$$

Proof: Let $\omega = g\, dz$. By definition, $\mathrm{Res}_a\, f\omega$ is the coefficient of the $1/(z-a)$ term in the Laurent series $l = f_a\, g_a$, where the product is the Cauchy product of the Laurent series. By assumption, the lowest power of $(z-a)$ appearing in g_a with a nonzero coefficient is greater than or equal to n. Therefore the only terms of f_a that can contribute to the residue after multiplication by a term of g_a are the terms of order less than $-n$, i.e., the terms of $t_{-n}(f_a)$. □

Now we define a map

$$\mathrm{Res}\colon L^{(1)}(-D) \to t_D(\mathbf{P}_0^R)^*$$

as follows. Let $R = (T, A)$ be a compact Riemann surface. For each point p in T, choose a chart C_p centered at p. Define

$$\mathrm{Res}(\omega) = (\Pi \mapsto \sum_{p \in T} \mathrm{Res}_0\ \Pi(p)\omega_p). \tag{2}$$

In other words, for any one form ω on R with $(\omega) \geq D$, $\mathrm{Res}(\omega)$ is the linear map that takes a Laurent polynomial divisor Π with finite support and bounded by $-D$ to the sum over all points p of the residues at zero of the products $\Pi(p)\omega_p$. Note that the sum is finite because (a) ω has negative Laurent series terms at finitely many points and (b) Π has finite support.

Since $H^1(D) = t_D(\mathbf{P}_0^R)/t_D(M(R))$, the map (2) induces a map (1) if we have

$$\mathrm{Res}(\omega)(t_D(\Lambda_f)) = 0 \tag{3}$$

for all meromorphic functions f on R. We shall now show that (3) holds. By definition we have

$$\mathrm{Res}(\omega)(t_D(\Lambda_f)) = \sum_{p \in T} \mathrm{Res}_0\ t_{-D(p)}(f_{p\,0})\omega_p,$$

where $f_{p\,0}$ denotes the Laurent series expansion at zero of the local function f_p on the chart C_p centered at p. By assumption $\mathrm{ord}_0\ \omega_p \geq D(p)$ at every point p, so the conditions of the lemma are satisfied, and we have

$$\begin{aligned}\mathrm{Res}(\omega)(t_D(\Lambda_f)) &= \sum_{p \in T} \mathrm{Res}_0\ f_p\omega_p \\ &= \sum_{p \in T} \mathrm{Res}_0\ (f\omega)_p \\ &= \sum_{p \in T} \mathrm{Res}_p\ f\omega. \end{aligned} \tag{4}$$

But the residue theorem (§ 1.4) says exactly that the right-hand side of (4) is zero. This establishes (3), as required.

5.2. The Serre Duality Theorem

The Serre duality theorem says that the map Res defined in the previous section is an isomorphism:

Theorem (Serre Duality): Let R be a compact Riemann surface, and let D be an integer-valued divisor on R with finite support. Then the map

$$\mathrm{Res}\colon L^{(1)}(-D) \to H^1(D)^*$$

is an isomorphism of complex vector spaces.

For the proof, see [Miranda 1995], VI, Theorem 3.3.

Corollary: Let R and D be as in the statement of the theorem. Then $L^{(1)}(-D)$ and $H^1(D)$ are isomorphic as complex vector spaces.

Proof: $H^1(D)$ is a finite-dimensional complex vector space (§ 4.4), so it is isomorphic to $\mathbf{C}^n$, where $n = \dim H^1(D)$. $\mathbf{C}^n$ is isomorphic to $(\mathbf{C}^n)^*$ via the map

$$u \mapsto (v \mapsto u \cdot v).$$

Compare the discussion of linear products in *The General Derivative*, § 4.1. These facts, together with the theorem, establish the result. □

6. The Riemann-Roch Theorem

We now have all the theory we need to state and prove the Riemann-Roch theorem.

Theorem (Riemann-Roch): Let R be a compact Riemann surface of genus g that has at least one nonconstant meromorphic function. Let D be an integer-valued divisor on R with finite support, and let K be a canonical divisor on R. Then

$$\dim L(D) - \deg D = \dim L(K - D) - g + 1.$$

To prove the theorem, we need a lemma.

Lemma: Let R, D, and K be as in the statement of the theorem. Then $L^{(1)}(D)$ and $L(D+K)$ are isomorphic as complex vector spaces.

Proof: Let $K = (\omega)$, where ω is a meromorphic one form on R. Define a map $\mu_K : L(D+K) \to M^{(1)}(R)$ as follows:

$$\mu_K(f) = f\omega.$$

We have

$$\begin{aligned}(f\omega) + D &= (f) + (\omega) + D \\ &= (f) + K + D \\ &\geq 0,\end{aligned}$$

since f lies in $L(D+K)$. Therefore $f\omega$ lies in $L^{(1)}(D)$, i.e.,

$$\mu_K(L(D+K)) \subseteq L^{(1)}(D).$$

Now choose a meromorphic one form χ in $L^{(1)}(D)$. Because (ω) is defined, ω is not identically zero, and so by the proposition in § 3.3, there exists a unique meromorphic function f on R such that $\chi = f\omega$. We have

$$\begin{aligned}(f) + D + K &= (f) + D + (\omega) \\ &= (f\omega) + D \\ &= (\chi) + D \\ &\geq 0,\end{aligned}$$

since χ lies in $L^{(1)}(D)$. Therefore f lies in $L(D+K)$, and $\mu_K(f) = \chi$. This shows that

$$L^{(1)}(D) \subseteq \mu_K(L(D+K)).$$

Therefore the image of μ_K is $L^{(1)}(D)$. Further, μ_K is injective by the uniqueness of f and it is linear, so it induces an isomorphism between its domain and its image. □

Proof of the theorem: By § 4.4, Proposition 2, it suffices to prove the following:

i. $\dim H^1(D) = \dim L(K-D)$.
ii. $\dim H^1(0) = g$.

(i) By the lemma, we have

$$\dim L(K-D) = \dim L^{(1)}(-D). \tag{1}$$

By the corollary to the Serre duality theorem (§ 5.2), we have

$$\dim L^{(1)}(-D) = \dim H^1(D). \tag{2}$$

(1) and (2) establish (i).

(ii) By Corollary 1 of § 3.5, we have

$$\deg K = 2g - 2. \tag{3}$$

By (i) we have

$$\dim H^1(K) = \dim L(K-K) = \dim L(0) = 1 \tag{4}$$

and

$$\dim H^1(0) = \dim L(K-0) = \dim L(K). \tag{5}$$

By Proposition 2 of § 4.4 with $D = K$, we have

$$\dim L(K) - \deg K = \dim H^1(K) - \dim H^1(0) + 1. \tag{6}$$

Substituting (3), (4), and (5) into (6) yields

$$\dim H^1(0) - (2g-2) = 1 - \dim H^1(0) + 1. \tag{7}$$

Solving for $\dim H^1(0)$ in (7) yields (ii). □

Corollary: Let R, D, and K be as in the statement of the theorem. Then

$$\dim H^1(0) = \dim L^{(1)}(0) = \dim L(K) = g.$$

Proof: In the proof of the theorem, we showed that $\dim L(K) = \dim H^1(0)$ and $\dim H^1(0) = g$. From the lemma, we have $\dim L^{(1)}(0) = \dim L(0+K) = \dim L(K)$. □

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