Integration in Real Vector Spaces

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Revised August 2021

The subject of this paper is integration in real vector spaces of finite dimension. Starting with the one-variable integral learned in first-year calculus, this paper connects that case to the more general theory. It then discusses path integrals and their extensions to higher dimensions. It presents the basic theory of differential forms.

This paper can serve as a bridge between integration as taught in first- and second-year calculus and more advanced material, e.g., differential forms as taught in undergraduate or graduate real analysis and differential geometry. I recommend that you read my paper *The General Derivative*, at least through § 7, before reading this paper. Otherwise, the prerequisites for reading this paper are the same as for reading that one.

Through out this paper we consider mappings $f: X \to Y$, where X and Y are finite-dimensional real vector spaces.

1. Integration in R

First-year calculus: We begin with the case $X = Y = \mathbf{R}$. This is the case of a single real variable, covered in first-year calculus. Let $f: \mathbf{R} \to \mathbf{R}$ be a function defined and continuous on a closed interval $s = [a, b]$.[1] In first-year calculus, we learn that we can write the formula

$$\int f(x)\, dx. \tag{1}$$

This formula makes sense if and only if there exists a function $F: \mathbf{R} \to \mathbf{R}$, defined and differentiable on s, such that $F'(x) = f(x)$ everywhere on s, where $F'(x)$ is the derivative of $F(x)$. In this case we write

$$\int f(x)\, dx = F(x), \tag{2}$$

where the equality is taken modulo the relation that two functions are equivalent if they differ by a constant on s. In other words, $F(x)$ is a representative of the class $\{\ G: \mathbf{R} \to \mathbf{R} \mid \text{for some } C \in \mathbf{R} \text{ and all } x \in s, G(x) = F(x) + C\ \}$.

We also learn that the formula

$$\int_a^b f(x)\, dx \tag{3}$$

represents the area under the curve from a to b, represented as a limit of sums of rectangles.

Finally, we learn that formulas (2) and (3) are related in the following way: if (2) holds, then we may compute (3) according to the formula

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{4}$$

Here F represents an element of the class $\int f(x)\, dx$; the particular one doesn't matter, because the constant term cancels out.

Formula (1) is usually called an **indefinite integral**. We may also call it a **formal integral**, because it maps one function to another without computing any numeric result. Formula (2) expresses the fact that the formal integral is an antiderivative. Formula (3) is usually called a **definite integral**. Formula (4), which expresses the relationship

[1] An **closed interval** is a set of real numbers lying between two numbers, including the endpoints. For example, the set of all real numbers r such that $-1 \le r \le 1$ is a closed interval. We often write closed intervals by listing the endpoints in brackets, e.g., $[-1, 1]$. An **open interval** is similar, but it excludes its endpoints and is written with parentheses. For example, the open interval $(-1, 1)$ represents all real numbers r such that $-1 < r < 1$.

between the indefinite integral and the definite integral, is one way of stating the **fundamental theorem of calculus**.

The linear map dx: In first-year calculus, the symbol dx is mainly a marker that indicates the variable of integration. However, there is a deeper theory here: by interpreting the symbol dx as a linear map, we can connect the single-variable integral to the general derivative in the following elegant way.

Let the symbol dx mean the identity map from $\mathbf{R}$ to $\mathbf{R}$, i.e., the map $h \mapsto h$, for all h in $\mathbf{R}$. Then dx is a linear map. Further, by the rules for multiplying numbers by functions, $f(x)\,dx$ represents the linear map $h \mapsto f(x)h$. If $f(x) = F'(x)$, then $f(x)\,dx = h \mapsto F'(x)h$ is our old friend $DF(x)$, where D is the differential operator defined in *The General Derivative*. With this interpretation, formula (2) becomes

$$\int DF(x) = F(x). \tag{5}$$

In the context of integration, we write d instead of D for the differential operator. Also, we often write just f instead of $f(x)$, if the variable of the function domain X is either obvious or unimportant. Therefore, (5) often appears in the alternate form

$$\int df = f.$$

This formula elegantly expresses the relationship between the derivative and the integral. In *The General Derivative* we wrote

$$\int DF(x)(1)\,dx$$

in order to convert the linear map $DF(x)$ to a real number so that we could integrate it. However, this really was not necessary: by interpreting dx as a linear map, we can instead write

$$\int DF(x),$$

which is equivalent and simpler.

Differential forms: The concept of a **differential form** generalizes the expressions $f(x)\,dx$ that appear in one-dimensional integrals; using differential forms, one can integrate over domains of zero or more dimensions. We use the symbol ω to denote a differential form. Every differential form has a **degree** corresponding to the dimension of the domain of integration: a point, a one-dimensional path, a two-dimensional surface, etc.

Zero forms: When $X = Y = \mathbf{R}$, a differential form of degree zero, also called a **zero form**, is a function $f\colon \mathbf{R} \to \mathbf{R}$. Let $\Omega^0(\mathbf{R})$ denote the set of zero forms. It is a real vector space, according to the rules $rf = x \mapsto rf(x)$ and $f_1 + f_2 = x \mapsto f_1(x) + f_2(x)$.

We integrate a zero form over a set of dimension zero, i.e., a point. To integrate the zero form f at a point a, we evaluate f at a. That is,

$$\int_a f = f(a),$$

where a is a real number.

One forms: When $X = Y = \mathbf{R}$, a differential form of degree one, also called a **one form**, is a map $\omega\colon \mathbf{R} \to L(\mathbf{R}, \mathbf{R})$, where as usual $L(\mathbf{R}, \mathbf{R})$ is the vector space of linear maps $\lambda\colon \mathbf{R} \to \mathbf{R}$. This space is often called the **dual space** of $\mathbf{R}$ and written $\mathbf{R}^*$. We may write any such one form as $f\,dx$, where f is a zero form and dx is the map $h \mapsto h$ discussed above. For example, if $\omega(x) = f(x)\,dx$ and $f(x) = x + 3$, then $\omega(2) = 5\,dx$, which is the linear map $M(5) = h \mapsto 5h$.

Let $\Omega^1(\mathbf{R})$ denote the set of one forms. It is a real vector space, according to the rules $r(f\,dx) = (rf)\,dx$ and $f_1\,dx + f_2\,dx = (f_1 + f_2)\,dx$.

Notice the following:

1. If the function f is differentiable, then we may apply the differential operator d to the zero form f to obtain the one form $df = f'\,dx$. The differential operator causes the degree of the form to go up by one from zero to one.

2. If the function f is integrable, then we may apply the integral operator $\int$ to the one form $f\ dx$ to obtain the zero form $\int f\ dx = F$, where $F' = f$. Again the equality is taken modulo the constant functions. The integral operator causes the degree of the form to go down by one from one to zero.

The operator d is called the **exterior derivative**. When applied to a zero form f, df just means the derivative Df. When applied to forms of higher degree, d and D have different meanings. For example, $DDf = D^2 f$ means the second derivative of f, whereas $ddf = 0$. We will explain this further below.

Let $\omega = f\ dx$ be a differential form such that f is integrable, so that $\omega = dF$ for some function F. In this case we say that ω is an **exact** differential form. When $d\omega = 0$, we say that ω is **closed**.

When $X = Y = \mathbf{R}$, all forms of degree two and greater are identically zero. Below we will explain why this is true; for now we will just accept it as a fact.

Closed intervals: We integrate a one form over a collection of points with dimension one. The simplest such collection is a closed interval $[a, b]$. By convention we assume $a < b$. Just as in first-year calculus, we integrate an exact one form over a closed interval $s = [a, b]$ by computing the definite integral from a to b:

$$\int_s f\ dx = \int_a^b f\ dx = F(b) - F(a),$$

where $dF = f\ dx$.

We often think of integration over a closed interval as the area under the curve $f(x)$ between a and b. It can also represent, for example, the work done by a particle moving from a to b with constant speed one and constant direction as it is acted on by the force f.

Paths: A **path** is a differentiable function $\sigma\colon s \to \mathbf{R}$, where $s = [a, b]$ is a closed interval. Integration over a path σ generalizes the idea of integration over a closed interval s. For example, a path $\sigma(t)$ can represent the position of a moving particle p as a function of time. Then integration over $\sigma(t)$ can represent the work done by p as it moves from $\sigma(a)$ to $\sigma(b)$ with varying speed.

We say that a path $\sigma\colon (a, b) \to \mathbf{R}$ is **closed** if its start and end points are the same, i.e., $\sigma(a) = \sigma(b)$. An example of a closed path in $\mathbf{R}$ is the motion of a particle that starts moving at some nonzero speed in one direction, slows to a stop as it moves, and starts moving faster and faster in the opposite direction until it reaches the point at which it started. We describe the slowing, stopping, and starting to emphasize that the path position must be a differentiable function of time.

To compute the integral of a one form over a path, we convert it to an integral over a closed interval, according to the following rule:

$$\int_\sigma f(x)\ dx = \int_s f(\sigma(t))\sigma'(t)\ dt. \tag{6}$$

If we write $f(x)\ dx = \omega(x)$, then $f(\sigma(t))\ dx = \omega(\sigma(t))$. From the discussion above, we can write $\sigma'(t)\ dt = d\sigma(t)$. Then remembering that $\omega(\sigma(t))$ and $d\sigma(t)$ are both linear maps from $\mathbf{R}$ to $\mathbf{R}$, we can restate the path integration rule (6) as follows:

$$\int_\sigma f(x)\ dx = \int_s \omega(\sigma(t)) \circ d\sigma(t). \tag{7}$$

The composition in (7) represents the map

$$h \mapsto (f(\sigma(t))\ dx)(d\sigma(t)(h)) = (f(\sigma(t)\ dx)(\sigma'(t)h) = f(\sigma(t))\sigma'(t)h.$$

We call this map the **pullback** of the differential form ω with respect to the path σ, because it uses the map σ from the line segment to $\mathbf{R}$ to pull the integral back from $\mathbf{R}$ to the line segment. We define the pullback $\sigma^*\omega$ according to the rule

$$(\sigma^*\omega)(t) = \omega(\sigma(t)) \circ d\sigma(t). \tag{8}$$

Using this definition of the pullback, we may restate the path integration rule more succinctly as follows:

$$\int_\sigma \omega = \int_s \sigma^* \omega. \tag{9}$$

This formulation readily generalizes to higher dimensions; we will discuss this further below.

Note that a closed interval is also a path: it is the special case where $\sigma(t) = t$ and $\sigma^* \omega = \omega$.

Regions: We generalize the notion of a path as follows: A **region** σ is a function from s to $\mathbf{R}$, where s is either a point or a closed interval. If s is a closed interval, then the function must be differentiable. This definition lets us use the same symbol σ to denote regions of dimension zero (i.e., points) and regions of dimension one (i.e., paths). Below we will discuss regions of higher dimension.

Formal linear combinations: A **formal linear combination** of elements is a sum of numbers times the elements, where we specify nothing about the elements except that we can combine them in this way. We collect syntactically identical terms as in high school algebra. For example, a formal linear combination of the elements x and y is a sum $ax + by$, for numbers a and b. Collecting terms means, for example, that $x + x = 2x$.

Chains: A **chain** is a finite collection of regions of the same dimension, together with an associated integer **weight** for each region. For each region σ in the chain, the associated weight n represents an orientation (via the sign of n) and a multiplicity (via the magnitude of n). The orientation says whether σ contributes positively or negatively to the chain. The multiplicity says how many times to count σ in the chain.

We use the symbol γ to represent a chain, and we write a chain as a formal linear combination of paths. For example, the one-dimensional chain γ consisting of path σ_1 with integer 2 and path σ_2 with integer -1 might be written

$$\gamma = 2\sigma_1 - \sigma_2.$$

A chain γ is **closed** if all of the regions appearing in γ are closed.

To integrate a form ω of degree k over a chain γ of dimension k, we integrate ω over each of the regions in γ, multiply by the integers, and add the results. For example:

$$\int_{2\sigma_1 - \sigma_2} \omega = 2 \int_{\sigma_1} \omega - \int_{\sigma_2} \omega.$$

Stokes' theorem for closed intervals: We can use integration over chains to express the fundamental theorem of calculus in an elegant way. Let $s = [a, b]$ be a closed interval. We can think of the points a and b as forming the boundary of s. So we define the **boundary chain** ∂s to be the zero-dimensional chain $b - a$. Then by the definition of integration over chains, and by the fundamental theorem of calculus, we have

$$\int_s df = \int_{\partial s} f,$$

because both sides are equal to $f(b) - f(a)$. This statement is the simplest case of a more general theorem called the generalized Stokes' theorem. We will state and prove the general theorem later in this paper.

Stokes' theorem for paths: We can easily extend Stokes' theorem to path integrals. Indeed, fix $s = [a, b]$ and $\sigma \colon s \to \mathbf{R}$. Then by definition we have

$$\int_\sigma df = \int_s \sigma^* df = \int_s df(\sigma(t)) \circ d\sigma(t)$$

By the chain rule, this is

$$\int_s d(f \circ \sigma)(t) = f(\sigma(b)) - f(\sigma(a)).$$

Now define the boundary chain $\partial\sigma = \sigma(b) - \sigma(a)$. Then by definition we have

$$\int_\sigma df = \int_{\partial\sigma} f.$$

Again we will extend this statement to higher dimensions below.

Path independence for exact one forms: Notice that the boundary chain $\partial\sigma$ depends only on the endpoints $\sigma(a)$ and $\sigma(b)$ of the path σ and not on the path itself: for any two paths $\sigma_1 \colon s \to \mathbf{R}$ and $\sigma_2 \colon s \to \mathbf{R}$ with $\sigma_1(a) = \sigma_2(a)$ and $\sigma_1(b) = \sigma_2(b)$, we have $\partial\sigma_1 = \partial\sigma_2$. Therefore by Stokes' theorem, for any exact one form df, if σ_1 and σ_2 have the same endpoints, then

$$\int_{\sigma_1} df = \int_{\sigma_2} df.$$

In this case we say that the integral is **independent of the path** (but it does depend on the path endpoints). For example, the work done by a particle moving from point a to b and acted on by force df is $f(b) - f(a)$ regardless of how fast the particle moves or whether it speeds up, slows down, stops, or changes direction.

A related consequence of Stokes' theorem is that for any closed path σ, we have $f(\sigma(b)) - f(\sigma(a)) = 0$, and so

$$\int_{\sigma} df = 0.$$

Similarly, for any closed chain γ, we have

$$\int_{\gamma} df = 0.$$

Integrating non-exact one forms: If the one form ω is not exact, then we cannot evaluate the definite integral $\int_a^b \omega$. However, it may be the case that for one or more paths $\sigma \colon (a, b) \to \mathbf{R}$, the one form $\sigma^* \omega$ is exact. In this case we can evaluate the integral $\int_{\sigma} \omega$ for any such path. Further, the value of the integral may depend on the path.

The most beautiful example of this phenomenon that I know of comes from complex analysis. Of course it is cheating a bit to present an example from complex analysis in a paper about real integration. However, I think it is worth taking a minor detour to study this example. Also, this example shows how path integration in complex analysis is closely related to path integration in real analysis.

If you haven't studied complex analysis, don't worry. All you need to know to understand this example is the following:

1. The **complex numbers C** are the real numbers **R** together with a number i such that $i^2 = -1$. Multiples of i such as i, $-i$, $2i$, etc. are called **pure imaginary numbers**.
2. In general, a complex number z is the sum of a real number and a pure imaginary number, i.e., $z = a + bi$ for some real numbers a and b. The complex numbers form a plane, with the real numbers as the horizontal axis and the pure imaginary numbers as the vertical axis.
3. Every complex number z lying on the unit circle centered at the origin may be written $z(t) = \cos t + i \sin t$, where t is an angle in radians from the positive real axis. We define $e^{it} = \cos t + i \sin t$.
4. A **complex function** is a mapping $f \colon \mathbf{C} \to \mathbf{C}$. We can take derivatives and integrals of complex functions, similarly to the corresponding operations for real functions. The derivative of $z(t) = \cos t + i \sin t = e^{it}$ with respect to t is $z'(t) = -\sin t + i \cos t = ie^{it}$.

Now let s be the closed interval $[0, 2\pi]$, let $\sigma \colon s \to \mathbf{C}$ be the complex path $\sigma(t) = e^{it}$, and let us compute the integral

$$\int_{\sigma} \frac{1}{z}\, dz.$$

By definition this is

$$\int_0^{2\pi} \frac{1}{\sigma(t)}\, \sigma'(t)\, dt = \int_0^{2\pi} \frac{1}{e^{it}}\, ie^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

Notice that σ is a closed path, but the value of the integral is nonzero. Also, the value of the integral depends on the path. For example, let $\sigma_2 \colon s \to \mathbf{C}$ be the complex path $\sigma_2(t) = 2e^{it}$. Then $\int_{\sigma_2} \frac{1}{z}\, dz = 4\pi i$. However, σ and σ_2 have

the same endpoints, because $e^0 = e^{2\pi i} = e^{4\pi i} = 1$. More generally, for any path σ_n that winds n times around the unit circle, the value of the path integral is $2\pi i n$. This computation shows that the complex differential form $\frac{1}{z}\,dz$ is not exact, i.e., there is no complex function f defined on the unit circle in the complex plane such that $df = \frac{1}{z}\,dz$.

2. Integration over Points and Paths in $\mathbf{R}^n$

In this section we let $X = \mathbf{R}^n$ and $Y = \mathbf{R}$. We consider functions $f\colon \mathbf{R}^n \to \mathbf{R}$, i.e., functions $f(x)$ where $x = (x_1, \ldots, x_n)$ is an element of $\mathbf{R}^n$. We continue to integrate zero forms over points and one forms over paths.

Differential forms: A zero form is a function $f\colon \mathbf{R}^n \to \mathbf{R}$. A one form is a map $\omega\colon \mathbf{R}^n \to L(\mathbf{R}^n, \mathbf{R})$. The space $L(\mathbf{R}^n, \mathbf{R})$ is often called the **dual space** of $\mathbf{R}^n$ and written $(\mathbf{R}^n)^*$. These formulas are the same as in the previous section, except that now we have $X = \mathbf{R}^n$ instead of $X = \mathbf{R}$.

To integrate a one form ω in the case $X = \mathbf{R}^n$, we have to express ω in terms of the coordinates $x_1, \ldots, x_n$. To do that, we make the following construction. Let $dx_i\colon \mathbf{R}^n \to \mathbf{R}$ be the linear map that projects onto coordinate i of $\mathbf{R}^i$. That is, if $h = (h_1, \ldots, h_n)$ is an element of $\mathbf{R}^n$, then $dx_i(h) = h_i$. Let $f_i\colon \mathbf{R}^n \to \mathbf{R}$ be functions, and write

$$f_1\ dx_1 + \cdots + f_n\ dx_n \tag{1}$$

to mean the map $\lambda\colon \mathbf{R}^n \to L(\mathbf{R}^n, \mathbf{R})$ given by

$$\lambda(x)(h) = f_1(x)\ dx_1(h) + \cdots + f_n(x)\ dx_n(h)$$

$$= f_1(x)h_1 + \cdots + f_n(x)h_n.$$

Then we may write every one form ω in the manner of (1), with a suitable choice of functions f_i. We make the set of one forms $\Omega^1(\mathbf{R}^n)$ into a vector space by distributing scalar multiplication over addition and by collecting additive terms in the obvious way.

By definition, a one form ω is exact if and only if there exists a function $f\colon \mathbf{R}^n \to \mathbf{R}$ such that $df = \omega$. By the definition of the differential operator d, this is true if and only if, for all i, $f_i = D_i f$, where D_i is the partial derivative with respect to the coordinate x_i. In this case we can formally integrate the one form, yielding f:

$$\int D_1 f\ dx_1 + \cdots + D_n f\ dx_n = \int df = f.$$

As usual, the right-hand equality is modulo the constant functions.

Integration over points and paths: When $X = \mathbf{R}^n$ and $Y = \mathbf{R}$, integration of a zero form f over a point a is the same as before:

$$\int_a f = f(a).$$

When $X = \mathbf{R}^n$ and $Y = \mathbf{R}$, a path σ is a differentiable mapping from a closed interval $s = [a, b]$ to $\mathbf{R}^n$. For example, let $n = 2$, $a = 0$, $b = 1$, and $\sigma(t) = (t, t)$. This path traces the diagonal line from $(0, 0)$ to $(1, 1)$ in $\mathbf{R}^2$ as t goes from 0 to 1.

To integrate a one form over a path σ, we use the same definition as for $X = \mathbf{R}$, i.e.,

$$\int_\sigma \omega = \int_s \sigma^* \omega.$$

The definition of the map $\sigma^*\omega$ given in the previous section goes through when $X = \mathbf{R}^n$.

As an example with $n = 2$, let $\omega = f_1\ dx_1 + f_2\ dx_2$ and $\sigma(t) = (\sigma_1(t), \sigma_2(t))$. Then the definition of $\sigma^*\omega$ gives

$$\int_\sigma \omega = \int_s \omega(\sigma(t)) \circ d\sigma(t). \tag{2}$$

The composition in (2) represents the map

$$h \mapsto (f_1(\sigma(t))\ dx_1 + f_2(\sigma(t))\ dx_2)((\sigma_1'(t)h, \sigma_2'(t)h)) = f_1(\sigma(t))\sigma_1'(t)h + f_2(\sigma(t))\sigma_2'(t)h.$$

Therefore we have

$$\int_\sigma \omega = \int_s f_1(\sigma(t))\sigma_1'(t)\ dt + f_2(\sigma(t))\sigma_2'(t)\ dt. \tag{3}$$

If we write $f = (f_1, f_2)$ and $dx = (dx_1, dx_2)$, then we can write $\omega = f_1\ dx_1 + f_2\ dx_2 = f \cdot dx$, where where the dot represents the dot product. Further, if we write $d\sigma(t) = (\sigma_1'(t)\ dt, \sigma_2'(t)\ dt)$, then we can rewrite equation (3) as follows:

$$\int_\sigma \omega = \int_s f(\sigma(t)) \cdot d\sigma(t) = \int_s (f \circ \sigma) \cdot d\sigma. \tag{4}$$

Equation (4) has a physical or geometric interpretation: the dot product $f(\sigma(t)) \cdot d\sigma(t)$ represents the contribution at each point $\sigma(t)$ of the vector field f with respect to the instantaneous velocity vector $d\sigma(t)$ of the path σ. Integrating this expression with respect to t sums up all the contributions. For example, if $f(x)$ represents a force at each point x, and $\sigma(t)$ represents the position of a moving point mass at time t, then the path integral represents the total work done by the point mass in the time interval $[a, b]$.

Note that when $X = \mathbf{R}^n$ and $n > 1$, we cannot integrate a one form directly over a closed interval: we must use a path. This is because there are more coordinates in the domain (n) than the degree of the form (1). We will have more to say about this in the next section.

From the definition of the pullback σ, we can see that for any one forms ω_1 and ω_2, we have $\sigma^*(\omega_1 + \omega_2) = \sigma^*\omega_1 + \sigma^*\omega_2$. Therefore

$$\int_\sigma \omega_1 + \omega_2 = \int_s \sigma^*(\omega_1 + \omega_2) = \int_s \sigma^*\omega_1 + \sigma^*\omega_2 = \int_s \sigma^*\omega_1 + \int_s \sigma^*\omega_2 = \int_\sigma \omega_1 + \int_\sigma \omega_2.$$

Thus integration over paths is well-behaved with respect to sums of differential forms. For example,

$$\int_\sigma f_1\ dx_1 + f_2\ dx_2 = \int_\sigma f_1\ dx_1 + \int_\sigma f_2\ dx_2.$$

Stokes' theorem: Stokes' theorem for paths applies in $\mathbf{R}^n$ for any $n > 0$. Nothing in the argument given in the previous section depends on dimension one.

Path dependence: The observations about path independence for exact one forms and integration of non-exact one forms carry over identically from the previous section into this one. Again, nothing depends on the dimensions of X.

3. Higher-Dimensional Regions

Now we turn our attention to regions of dimension $k \geq 2$. We let $X = \mathbf{R}^n$ for $n \geq 2$ and $Y = \mathbf{R}$.

3.1. Differential Forms

Two forms: To integrate over regions of dimension two, we use a differential form of degree two, also called a **two form**. A two form is a map $\omega\colon \mathbf{R}^n \to L(\mathbf{R}^n, \mathbf{R}) \wedge L(\mathbf{R}^n, \mathbf{R})$, where the expression to the right of the arrow denotes the **alternating product** or **wedge product** of the vector space $L(\mathbf{R}^n, \mathbf{R})$ with itself. This alternating product is the vector space of formal linear combinations of vectors $\lambda_1 \wedge \lambda_2$, where λ_1 and λ_2 represent linear maps in $L(\mathbf{R}^n, \mathbf{R})$, subject to the following rules:

Alt-1. $\lambda_1 \wedge \lambda_2 = -\lambda_2 \wedge \lambda_1$.

Alt-2. $(\lambda_1 + \lambda_2) \wedge \lambda_3 = \lambda_1 \wedge \lambda_2 + \lambda_1 \wedge \lambda_3$.

Alt-3. For all $r \in \mathbf{R}$, $r(\lambda_1 \wedge \lambda_2) = (r\lambda_1) \wedge \lambda_2$.

These rules reflect the structure of two-dimensional integration. In particular, rule **Alt-1** reflects the change of sign that occurs when reversing the orientation of a boundary. We will discuss this issue further below. For now, we continue with the algebraic properties.

From **Alt-1** we can derive $\lambda \wedge \lambda = -\lambda \wedge \lambda$, so $2(\lambda \wedge \lambda) = 0$, so $\lambda \wedge \lambda = 0$. From **Alt-1** and **Alt-2** we can derive

$$\lambda_1 \wedge (\lambda_2 + \lambda_3) = -(\lambda_2 + \lambda_3) \wedge \lambda_1 = -(\lambda_2 \wedge \lambda_1 + \lambda_3 \wedge \lambda_1) = \lambda_1 \wedge \lambda_2 + \lambda_1 \wedge \lambda_3.$$

From **Alt-1** and **Alt-3** we can derive

$$r(\lambda_1 \wedge \lambda_2) = -r(\lambda_2 \wedge \lambda_1) = -(r\,\lambda_2) \wedge \lambda_1 = \lambda_1 \wedge (r\,\lambda_2).$$

The linear maps $dx_i = x \mapsto x_i$ $(1 \le i \le n)$ form a basis for $L(\mathbf{R}^n, \mathbf{R})$. Indeed, each map λ in $L(\mathbf{R}^n, \mathbf{R})$ is a dot product $x \mapsto v \cdot x$ for some vector $v = (v_1, \ldots, v_n)$ in $\mathbf{R}^n$, and so we may write λ as the linear combination $v_1\, dx_1 + \cdots + v_n\, dx_n$ of the basis vectors dx_i. Therefore the elements $dx_i \wedge dx_j$ $(1 \le i \le n, 1 \le j \le n, i < j)$ form a basis for $L(\mathbf{R}^n, \mathbf{R}) \wedge L(\mathbf{R}^n, \mathbf{R})$. We require $i < j$ to eliminate duplicates according to rules **Alt-1** and **Alt-2**. For example, $\mathbf{R}^2 \wedge \mathbf{R}^2$ has the single basis vector $dx_1 \wedge dx_2$, and $\mathbf{R}^3 \wedge \mathbf{R}^3$ has basis vectors $dx_1 \wedge dx_2$, $dx_1 \wedge dx_3$, and $dx_2 \wedge dx_3$. Other combinations are redundant. For example, $dx_1 \wedge dx_1 = 0$, and $dx_2 \wedge dx_1 = -dx_1 \wedge dx_2$. In general $L(\mathbf{R}^n, \mathbf{R})$ has dimension n, and the number of basis vectors $dx_i \wedge dx_j$ (i.e., the dimension of $L(\mathbf{R}^n, \mathbf{R}) \wedge L(\mathbf{R}^n, \mathbf{R})$) is $m = \binom{n}{2} = \dfrac{n(n-1)}{2}$.

Let $e_1, \ldots, e_m$ denote the basis vectors $dx_i \wedge dx_j$, constructed as described above and arranged by increasing order of the sums $i + j$. Because the vectors e_i are a basis for $L(\mathbf{R}^n, \mathbf{R})$, and a two form is a map $\omega\colon \mathbf{R}^n \to L(\mathbf{R}^n, \mathbf{R}) \wedge L(\mathbf{R}^n, \mathbf{R})$, we can write any two form $\omega(x)$ as a sum

$$\omega(x) = f_1(x)\, e_1 + \cdots + f_m(x)\, e_m,$$

where each f_i is a function from $\mathbf{R}^n$ to $\mathbf{R}$ that gives the component of ω along e_i at each point x. For example, when $n = 3$, we have

$$\omega(x) = f_1(x)\, dx_1 \wedge dx_2 + f_2(x)\, dx_1 \wedge dx_3 + f_3(x)\, dx_2 \wedge dx_3.$$

If a is a point in $\mathbf{R}^3$, such that $f_i(a) = a_i$ for each i, then evaluating ω at a yields

$$\omega(a) = a_1\, dx_1 \wedge dx_2 + a_2\, dx_1 \wedge dx_3 + a_3\, dx_2 \wedge dx_3,$$

where the a_i are numbers. This notation looks ambiguous: does $a_1\, dx_1 \wedge dx_2$ mean $a_1\, (dx_1 \wedge dx_2)$ or $(a_1\, dx_1) \wedge dx_2$? However, by rule **Alt-3**, it doesn't matter: both expressions represent the same element of $L(\mathbf{R}^3, \mathbf{R}) \wedge L(\mathbf{R}^3, \mathbf{R})$.

k **forms:** To generalize to regions of dimension k, we use a differential form of degree k, also called a k **form**. It is a map $\omega\colon \mathbf{R}^n \to \wedge^k L(\mathbf{R}^n, \mathbf{R})$, where the expression to the right of the arrow denotes the k-fold alternating product of $L(\mathbf{R}^n, \mathbf{R})$ with itself. This definition agrees with the definitions already given for zero forms, one forms, and two forms, if we interpret $\wedge^0 L(\mathbf{R}^n, \mathbf{R})$ as $\mathbf{R}$ and $\wedge^1 L(\mathbf{R}^n, \mathbf{R})$ as $L(\mathbf{R}^n, \mathbf{R})$. For $k > 1$, the k-fold alternating product is the real vector space whose elements are formal linear combinations of vectors $v = \lambda_1 \wedge \cdots \wedge \lambda_k$, where each λ_i is an element of $L(\mathbf{R}^n, \mathbf{R})$, subject to the following rules:

Alt-1. Exchanging any two adjacent elements λ_i and λ_{i+1} of v yields $-v$.

Alt-2. $(\lambda_{11} + \lambda_{12}) \wedge v = \lambda_{11} \wedge v + \lambda_{12} \wedge v$, where $v = \lambda_2 \wedge \cdots \wedge \lambda_k$.

Alt-3. For all $r \in \mathbf{R}$, $r(\lambda_1 \wedge v) = (r\,\lambda_1) \wedge v$, where $v = \lambda_2 \wedge \cdots \wedge \lambda_k$.

These rules generalize the rules we gave for the $k = 2$ case.

From **Alt-1**, we conclude that if any two distinct elements λ_i and λ_j of v are equal, then $v = 0$. Using **Alt-1**, we can apply **Alt-2** at any element λ_i. For example:

$$\lambda_1 \wedge \lambda_2 \wedge (\lambda_{31} + \lambda_{32}) = -(\lambda_1 \wedge (\lambda_{31} + \lambda_{32}) \wedge \lambda_2) = (\lambda_{31} + \lambda_{32}) \wedge \lambda_1 \wedge \lambda_2 = \lambda_{31} \wedge \lambda_1 \wedge \lambda_2 + \lambda_{32} \wedge \lambda_1 \wedge \lambda_2$$

$$= -(\lambda_1 \wedge \lambda_{31} \wedge \lambda_2) - (\lambda_1 \wedge \lambda_{32} \wedge \lambda_2) = \lambda_1 \wedge \lambda_2 \wedge \lambda_{31} + \lambda_1 \wedge \lambda_2 \wedge \lambda_{32}.$$

From **Alt-1** and **Alt-3**, we conclude that multiplying v by r is equivalent to multiplying any element λ_i by r. For example:

$$r(\lambda_1 \wedge \lambda_2 \wedge \lambda_3) = -r(\lambda_1 \wedge \lambda_3 \wedge \lambda_2) = r(\lambda_3 \wedge \lambda_1 \wedge \lambda_2) = (r\,\lambda_3) \wedge \lambda_1 \wedge \lambda_2 = -\lambda_1 \wedge (r\,\lambda_3) \wedge \lambda_2. = \lambda_1 \wedge \lambda_2 \wedge (r\,\lambda_3).$$

For $k > 0$, let I be a strictly increasing function from the integers $[1, k]$ to the integers $[1, n]$. There are $m = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ such functions; arrange them in increasing order according to the sums $I(1) + \cdots + I(k)$, and call them $I_1, \ldots, I_m$. Let $e_i = dx_{I_i(1)} \wedge \cdots \wedge dx_{I_i(k)}$. Then the set $e_1, \ldots, e_m$ is a basis for $\wedge^k L(\mathbf{R}^n, \mathbf{R})$, and we can write any k form $\omega(x)$ as a sum

$$\omega(x) = f_1(x)\ e_1 + \cdots + f_m(x)\ e_m,$$

where each f_i is a function from $\mathbf{R}^n$ to $\mathbf{R}$. For example, when $n = 4$ and $k = 3$, we have $m = \binom{4}{3} = 4$, and the basis vectors are $e_1 = dx_1 \wedge dx_2 \wedge dx_3$, $e_2 = dx_1 \wedge dx_2 \wedge dx_4$, $e_3 = dx_1 \wedge dx_3 \wedge dx_4$, and $e_4 = dx_2 \wedge dx_3 \wedge dx_4$.

We write $\Omega^k(\mathbf{R}^n)$ to denote the set of k forms in $\mathbf{R}^n$. It is a vector space, according to the usual rules

$$\omega_1 + \omega_2 = x \mapsto \omega_1(x) + \omega_2(x)$$

$$r\omega = x \mapsto r(\omega(x))$$

In § 1, we mentioned that when $X = \mathbf{R}$, all forms of degree two and higher are identically zero. We can see this is true because $dx \wedge dx = 0$. In general, $\Omega^k(\mathbf{R}^n) = 0$ when $k > n$.

If you find yourself forgetting what n and k mean, just remember:

- n is the dimension of the domain space and the maximum number of symbols dx_i appearing in a form of any degree.
- k is the degree of a form (zero; or one more than the number of $\wedge$ symbols appearing in each term of the form). It is dimension of a region of integration, which is embedded in the domain space.

n and k are related only in that we must have $k \leq n$ to have a non-degenerate k form.

Primitive k forms: The representation of a k form in terms of the basis vectors e_i depends on the choice of coordinates x_j that provide the linear maps dx_j. We now discuss a useful alternative representation that is coordinate-free.

For $k > 0$, a vector in $\wedge^k L(\mathbf{R}^n, \mathbf{R})$ is a sum $\sum_i \lambda_{i1} \wedge \cdots \wedge \lambda_{ik}$. A k form $\omega(x)$ maps an element x in $\mathbf{R}^n$ to such a vector. Therefore we may represent ω as a sum of terms $\sum_i \omega_i$, where each term ω_i is a k form $\omega_i(x) = \lambda_{i1}(x) \wedge \cdots \wedge \lambda_{ik}(x)$, and the sum is taken in the vector space $\Omega^k(\mathbf{R}^n)$.

In this notation, each symbol $\lambda(x)$ denotes a linear map $h \mapsto \lambda(x)(h)$. We could write $\omega(x) = \bigwedge_{i=1}^{k} \lambda_i(x)(h)$, but then the notation gets cluttered. Just remember that there are actually two variables: the variable x that maps to $\lambda(x)$, and the variable h that maps to $\lambda(x)(h)$.

We will call a k form $\omega(x) = \lambda_1(x) \wedge \cdots \wedge \lambda_k(x)$ a **primitive k form**, to distinguish it from general k forms that are sums of primitive k forms. As an example, consider the k form $\omega(x) = f(x)\ dx_1 \wedge \cdots \wedge dx_k$. We may represent $\omega(x)$ as the primitive k form $(f(x)\ dx_1) \wedge \cdots \wedge dx_k$. The first element in the wedge product is the map $\lambda_1(x) = f(x)\ dx_1$. The other elements are constant maps $\lambda_i(x)$ such that $\lambda_i(x) = dx_i$ for all x.

Let $\mu(U, V)$ denote the vector space of maps from the vector space U to the vector space V. Then the representation of k forms in terms of primitive forms shows that

$$\Omega^k(\mathbf{R}^n) = \mu(\mathbf{R}^n, \bigwedge_{i=1}^{k} L(\mathbf{R}^n, \mathbf{R})) = \bigwedge_{i=1}^{k} \mu(\mathbf{R}^n, L(\mathbf{R}^n, \mathbf{R})).$$

In other words, a map to the alternating product is an alternating product of maps.

The wedge product of differential forms: Let k_1 and k_2 be integers greater than zero. Given a primitive k_1 form $\omega_1(x) = \lambda_{11}(x) \wedge \cdots \wedge \lambda_{1k_1}(x)$ and a primitive k_2 form $\omega_2(x) = \lambda_{21}(x) \wedge \cdots \wedge \lambda_{2k_2}(x)$. Let $k = k_1 + k_2$. We define the primitive k form $(\omega_1 \wedge \omega_2)(x)$ as follows:

$$(\omega_1 \wedge \omega_2)(x) = \lambda_{11}(x) \wedge \cdots \wedge \lambda_{1k_1}(x) \wedge \lambda_{21}(x) \wedge \cdots \wedge \lambda_{2k_2}(x).$$

On the right, once we apply x we have k elements in $L(\mathbf{R}^n, \mathbf{R})$, and we can construct their wedge product as shown to get an element of $\bigwedge_{i=1}^{k} L(\mathbf{R}^n, \mathbf{R})$.

We call the operation $\wedge$ the **wedge product** of the primitive forms ω_1 and ω_2. We extend the wedge product to all pairs of k_1 and k_2 forms by linearity, i.e., we specify

$$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3.$$

Note, for example, that

$$f\ dx_1 \wedge g\ dx_2 = (f\ dx_1) \wedge (g\ dx_2) = fg\ dx_1 \wedge dx_2,$$

where the multiplication fg occurs in the vector space of functions.

The exterior derivative: The exterior derivative is a linear map $d\colon \Omega^k(\mathbf{R}^n) \to \Omega^{k+1}(\mathbf{R}^n)$, for $k \geq 0$. When $k = 0$, the exterior derivative d is the derivative D. When $k > 0$, we define

$$d(f\ e) = df \wedge e, \tag{1}$$

where e is a basis vector of $\wedge^k L(\mathbf{R}^n, \mathbf{R})$, f is differentiable, and $\wedge$ represents the wedge product of differential forms just defined. We extend d by linearity to all points v of $\Omega^k(\mathbf{R}^n)$ where the component functions f_i of v along the basis vectors e_i are differentiable. That is, we let $d \sum_i f_i\ e_i = \sum_i d(f_i\ e_i)$.

As an example, let ω be the one form $f_1\ dx_1 + f_2\ dx_2 \in \Omega^1(\mathbf{R}^2)$, and let us compute the two form $d\omega$. By linearity, we have

$$d\omega = d(f_1\ dx_1) + d(f_2\ dx_2).$$

By (1), we have

$$d\omega = (D_1 f_1\ dx_1 + D_2 f_1\ dx_2) \wedge dx_1 + (D_1 f_2\ dx_1 + D_2 f_2\ dx_2) \wedge dx_2.$$

By the extended form of **Alt-2**, this is

$$D_1 f_1\ dx_1 \wedge dx_1 + D_2 f_1\ dx_2 \wedge dx_1 + D_1 f_2\ dx_1 \wedge dx_2 + D_2 f_2\ dx_2 \wedge dx_2.$$

By the extended form of **Alt-1**, the $dx_i \wedge dx_i$ terms drop out, and we find

$$d\omega = D_2 f_1\ dx_2 \wedge dx_1 + D_1 f_2\ dx_1 \wedge dx_2.$$

By **Alt-1** and linearity, we have

$$d\omega = (D_1 f_2 - D_2 f_1)\ dx_1 \wedge dx_2,$$

because we can replace $dx_2 \wedge dx_1$ with $-dx_1 \wedge dx_2$ and collect like terms.

If ω is exact, then by definition there exists a function f such that $f_1 = D_1 f$ and $f_2 = D_2 f$. Therefore

$$d\omega = (D_1 D_2 f - D_2 D_1 f)\ dx_1 \wedge dx_2,$$

and since the second-order partial derivatives D_{ij} are symmetric in i and j, we have $d\omega = ddf = 0$, i.e., ω is closed.

In fact, it is true for any k form ω that if ω is exact and $d\omega$ exists, then ω is closed. To see this, first assume ω is a one form $df = \sum_i D_i f_i\ dx_i$. Then $ddf = \sum_{j,i} D_j D_i f_i\ dx_j \wedge dx_i$. Here i and j range over the n coordinates of $\mathbf{R}^n$. Each distinct pair (i, j) appears exactly twice in the sum, once as $D_i D_j f\ dx_i \wedge dx_j$ and once as $D_j D_i f\ dx_j \wedge dx_i$. Since the double partial derivatives are symmetric, everything cancels out to zero by **Alt-1**.

Now assume $\omega = d\omega_0$, for a k form ω_0 with $k > 0$, and write ω_0 in terms of the basis vectors e_i for $\Omega^k(\mathbf{R}^n)$:

$$\omega_0 = f_1\ e_1 + \cdots + f_m\ e_m.$$

Then we must have

$$\omega = d\omega_0 = df_1 \wedge e_1 + \cdots + df_m \wedge e_m.$$

It suffices to show that $d(df_i \wedge e_i) = 0$ for each i. But this is clear, because

$$df_i \wedge e_i = \sum_j D_j f_j\ dx_j \wedge e_i$$

and

$$d(df_i \wedge e_i) = \sum_{k,j} D_k D_j f_i\ dx_k \wedge dx_j \wedge e_i.$$

Again every pair (j, k) is represented exactly twice, once as $D_j D_k f\ dx_j \wedge dx_k \wedge e_i$ and once as $D_k D_j f\ dx_k \wedge dx_j \wedge e_i$, so everything cancels out to zero.

The product rule for the exterior derivative: Let f and g be functions from $\mathbf{R}^n$ to $\mathbf{R}$. As usual, let fg represent $x \mapsto f(x)g(x)$, where the product on the right-hand side is taken in $\mathbf{R}$. Then from the properties of the general

derivative, we have the product rule, i.e.,

$$d(fg) = (df)g + f(dg).$$

We now assert an analogous rule for the exterior derivative of a wedge product of forms. Let ω_1 be a k_1 form and ω_2 be a k_2 form, where $k_1 > 0$ and $k_2 > 0$. Then

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2. \tag{2}$$

Proof: By linearity, it suffices to show (2) in the case where $\omega_1 = f\ e_i$ and $\omega_2 = g\ e_j$, where e_i and e_j are basis vectors in $\Omega^{k_1}(\mathbf{R}^n)$ and $\Omega^{k_2}(\mathbf{R}^n)$, respectively. Then

$$d(\omega_1 \wedge \omega_2) = d(f\ e_i \wedge g\ e_j).$$

By rule **Alt-3**, we can move g across e_i:

$$= d(fg\ e_i \wedge e_j)$$

By the definition of the exterior derivative:

$$= d(fg) \wedge e_i \wedge e_j$$

By the product rule for functions:

$$= ((df)g + f(dg)) \wedge e_i \wedge e_j$$

By rules **Alt-2** and **Alt-3**:

$$= g\ df \wedge e_i \wedge e_j + f\ dg \wedge e_i \wedge e_j$$

By rule **Alt-3**, we can move the function g across $df \wedge e_i$:

$$= df \wedge e_i \wedge g\ e_j + f\ dg \wedge e_i \wedge e_j$$

By rule **Alt-1**, we can move the one form dg across e_i, but we have to multiply by -1 for each of the k_1 terms in e_i:

$$= df \wedge e_i \wedge g\ e_j + (-1)^{k_1} f\ e_i \wedge dg \wedge e_j$$

By the definition of ω_1 and ω_2:

$$= df \wedge e_i \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge dg \wedge e_j$$

By the definition of $d\omega_1$ and $d\omega_2$:

$$= d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2.$$

□

3.2. Integration

The formal integral: The formal integral is a linear map $\int : \Omega^{k+1}(\mathbf{R}^n) \to \Omega^k(\mathbf{R}^n)$, for $k \geq 0$. It is defined on the exact $k+1$ forms of $\Omega^{k+1}(\mathbf{R}^n)$ and maps $\omega = df$ to $\int \omega = \int df = f$.

Integrating two forms: We will call a k form ω **simple** if it is an element of the one-dimensional vector space $\Omega^k(\mathbf{R}^k)$. For example, $f(x)\ dx$ is a simple one form, and $f(x_1, x_2)\ dx_1 \wedge dx_2$ is a simple two form. Note that every simple k form is primitive, because the only basis vector in $\Omega^k(\mathbf{R}^k)$ is $dx_1 \wedge \cdots \wedge dx_k$.

In § 1, we integrated a simple one form over a closed interval $s = [a, b]$ in $\mathbf{R}$. To integrate a simple two form, we integrate over an ordered pair of closed intervals $s = (s_1, s_2) = ([a_1, b_1], [a_2, b_2])$, which we will call a **rectangle of dimension two**. In general, a rectangle of dimension k is an ordered k-tuple of closed intervals. A rectangle of dimension one is a closed interval. A rectangle of dimension zero is a point.

A rectangle of dimension two specifies a rectangular subset of $\mathbf{R}^2$, consisting of all points (x_1, x_2) in $\mathbf{R}^2$ such that x_i lies in the closed interval $[a_i, b_i]$ for each i. For example, if $s = ([0, 1], [0, 1])$, then s represents all points whose x_1 and x_2 coordinates both lie between zero and one.

Given a simple two form $\omega = f\ dx_1 \wedge dx_2$ and a rectangle $s = (s_1, s_2)$ of dimension two, we define the integral of ω over s as follows:

$$\int_s \omega = \int_s f\ dx_1 \wedge dx_2 = \int_{s_2} \int_{s_1} f\ dx_1 dx_2.$$

This notation means the following:

1. Interpret f as a function $f_1(x_1)\colon \mathbf{R} \to \mathbf{R}$, treating x_2 as a constant.
2. Integrate the simple one form $f_1(x_1)\ dx_1$ over the closed interval s_1, yielding a function $f_2(x_2)\colon \mathbf{R} \to \mathbf{R}$.
3. Integrate the simple one form $f_2(x_2)\ dx_2$ over the closed interval s_2, yielding a value in $\mathbf{R}$.

For example, let $\omega = x_1 x_2\ dx_1 \wedge dx_2$, and let $s = ([0, 1], [0, 1])$. Then

$$\int_s \omega = \int_0^1 \int_0^1 x_1 x_2\ dx_1 dx_2 = \int_0^1 \left[\frac{x_1^2 x_2}{2} \Big|_0^1 \right] dx_2 = \int_0^1 \frac{x_2}{2}\ dx_2 = \frac{x_2^2}{4} \Big|_0^1 = \frac{1}{4}.$$

This process is known as **double integration**.

To integrate a two form in $\Omega^2(\mathbf{R}^n)$, where $n \geq 2$, we fix a rectangle s of dimension two and construct a differentiable mapping $\sigma\colon s \to \mathbf{R}^n$. We will call the mapping σ a **two-dimensional region**. It is the two-dimensional analog of the one-dimensional path σ that we described in § 2. We make a rectangle s of dimension two into a two-dimensional region via the identity mapping $\sigma\colon s \to \mathbf{R}^2$.

Let ω be a two form in $\mathbf{R}^n$. We define the pullback $\sigma^* \omega$ as follows:

1. If $\omega(x) = \lambda_1(x) \wedge \lambda_2(x)$ is a primitive two form, then
$$(\sigma^* \omega)(y) = \sigma^*(\lambda_1(x)) \wedge \sigma^*(\lambda_2(x)) = (\lambda_1(\sigma(y)) \circ d\sigma(y)) \wedge (\lambda_2(\sigma(y)) \circ d\sigma(y)).$$
Here $\sigma^*(\lambda_i(x))$ is the same pullback for a one form that we defined in § 1 (8).
2. For any two forms ω_1 and ω_2, $\sigma^*(\omega_1 + \omega_2) = \sigma^* \omega_1 + \sigma^* \omega_2$.

When applied to combinations $\sum_i f_i\ e_i$ of the basis vectors e_i, these rules yield

1. $\sigma^*(f\ dx_i \wedge dx_j)(y) = f(\sigma(y))\ (dx_i \circ d\sigma(y)) \wedge (dx_j \circ d\sigma(y))$.
2. $\sigma^* \sum_i f_i\ e_i = \sum_i \sigma^*(f_i\ e_i)$.

Because the pullback $\sigma^* \omega$ is a simple two form defined on s, we can integrate it over s. We define

$$\int_\sigma \omega = \int_s \sigma^* \omega.$$

This is the same definition that we used for path integrals in § 1. The only difference is that we have now overloaded the symbol s to refer to a rectangle of dimension two and the symbol σ to represent a two-dimensional region. The overloading of symbols causes no confusion, so long as we remember the dimension k in which we are working.

As an example, let $\omega = f\ dx_1 \wedge dx_2 = x_1 x_2\ dx_1 \wedge dx_2$ and $s = ([0, 1], [0, 1])$ as in the previous example, and let $\sigma(y_1, y_2) = (2y_1, 2y_2)$. Let us compute $\int_\sigma \omega$. We have the following:

- $f(\sigma(y)) = 4 y_1 y_2$.
- $d\sigma(y) = (2dy_1, 2dy_2)$, i.e., $d\sigma(y)$ is the linear map $h \mapsto (2dy_1, 2dy_2)(h) = ((2dy_1)(h), (2dy_2)(h)) = (2h_1, 2h_2)$.
- $dx_1 \circ d\sigma(t) = 2dy_1$.
- $dx_2 \circ d\sigma(t) = 2dy_2$.

By definition we have

$$\int_\sigma \omega = \int_s \sigma^* \omega = \int_s \sigma^*(f\ dx_1 \wedge dx_2) = \int_s 4 y_1 y_2\ 2dy_1 \wedge 2dy_2 = 16 \int_s \omega.$$

By the previous example, the result is $\dfrac{16}{4} = 4$.

Integrating k forms: We now extend the integration of two forms to higher dimensions. To integrate a simple k form, we integrate over a rectangle s of dimension k. Let $s = (s_1, \ldots, s_k) = ([a_1, b_1], \ldots, [a_k, b_k])$. Then s specifies a rectangular subset of $\mathbf{R}^k$, consisting of all points $(x_1, \ldots, x_k)$ in $\mathbf{R}^k$ such that x_i lies in the closed interval $[a_i, b_i]$ for each i.

Given a simple k form $\omega = f\ dx_1 \wedge \cdots \wedge dx_k$ and a rectangle $s = (s_1, \ldots, s_k)$ of dimension $k > 2$, we define the integral of ω over s as follows:

$$\int_s \omega = \int_s f\ dx_1 \wedge \cdots \wedge dx_k = \int_{s_k} \left[\int_{(s_1, \ldots, s_{k-1})} f\ dx_1 \wedge \cdots \wedge dx_{k-1} \right] dx_k.$$

This notation means the following:

1. Interpret f as a function $f_{k-1}(x_1, \ldots, x_{k-1})\colon \mathbf{R}^{k-1} \to \mathbf{R}$, treating x_k as a constant.
2. Integrate the simple $k-1$ form $f_{k-1}(x_1, \ldots, x_{k-1})\ dx_1 \wedge \cdots \wedge dx_{k-1}$ over the rectangle $(s_1, \ldots, s_{k-1})$ of dimension $k-1$, yielding a function $f_k(x_k)\colon \mathbf{R} \to \mathbf{R}$. We do this by induction. In the base case, $k = 3$, so $k - 1 = 2$, and this is the integral of a simple two form as described in the previous subsection.
3. Integrate the simple one form $f_k(x_k)\ dx_k$ over the closed interval s_k, yielding a value in $\mathbf{R}$.

This process is known as **multiple integration**.

To integrate a k form in $\Omega^k(\mathbf{R}^n)$, where $n \geq k$, we fix a rectangle s of dimension k and construct a differentiable mapping $\sigma\colon s \to \mathbf{R}^n$. We will call the mapping σ a **k-dimensional region**. We make a rectangle s of dimension k into a k-dimensional region via the identity mapping $\sigma\colon s \to \mathbf{R}^k$.

Let ω be a k form. For $k = 0$ we define the pullback σ of a zero form $\omega = f$ as $(\sigma^* f)(y) = f(\sigma(y))$. For $k > 0$, we define the pullback $\sigma^* \omega$ as follows:

1. If $\omega(x) = \bigwedge_{i=1}^{k} \lambda_i(x)$ is a primitive k form, then $(\sigma^* \omega)(y) = \bigwedge_{i=1}^{k} \sigma^*(\lambda_i(x)) = \bigwedge_{i=1}^{k} \lambda_i(\sigma(y)) \circ d\sigma(y)$.
2. For any k forms ω_1 and ω_2, $\sigma^*(\omega_1 + \omega_2) = \sigma^* \omega_1 + \sigma^* \omega_2$.

From rule 1, it is clear that if ω_1 and ω_2 are primitive forms, then $\sigma^*(\omega_1 \wedge \omega_2) = \sigma^* \omega_1 \wedge \sigma^* \omega_2$. Then by linearity, this is true for all forms ω_1 and ω_2. We will use this fact later, when proving Stokes' theorem.

When applied to combinations $\sum_i f_i\ e_i$ of the basis vectors e_i, these rules yield

1. $\sigma^*(f\ dx_{i_1} \wedge \cdots \wedge dx_{i_k})(y) = f(\sigma(y))\ (dx_{i_1} \circ d\sigma(y)) \wedge \cdots \wedge (dx_{i_k} \circ d\sigma(y))$.
2. $\sigma^* \sum_i f_i\ e_i = \sum_i \sigma^*(f_i\ e_i)$.

The subscripts i_1 and i_k denote the fact that we have k distinct indices, but they may not lie in the range $[1, k]$ if $n > k$. For example, with $n = 3$ and $k = 2$, we could have $dx_1 \wedge dx_3$, so $i_1 = 1$ and $i_2 = 3$. These rules extend the rules given previously for the $k = 2$ case.

Because the pullback $\sigma^* \omega$ is a simple k form defined on s, we can integrate it over s. We define

$$\int_\sigma \omega = \int_s \sigma^* \omega.$$

This is the same definition that we used for the $k = 2$ case.

3.3. Stokes' Theorem

The oriented boundary chain in k dimensions: Let $s = (s_1, \ldots, s_k) = ([a_1, b_1], \ldots, [a_k, b_k])$ be a rectangle of dimension $k > 0$. We now define the **oriented boundary chain** ∂s of s. It is a chain of dimension $k - 1$. This is the higher-dimensional analog of the zero-dimensional boundary chain that we discussed in § 1.

For each i in $[1, k]$, let s/s_i represent the $(k-1)$-dimensional rectangle $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k)$ obtained by deleting s_i from s. For each $t \in \mathbf{R}$, let $\sigma_i(t)\colon s/s_i \to \mathbf{R}^k$ be the region of dimension $k-1$ that takes each point $(x_1, \ldots, x_{i-1}, x_i, \ldots, x_{k-1})$ in s/s_i to the point $(x_1, \ldots, x_{i-1}, t, x_i, \ldots, x_{k-1})$ in $\mathbf{R}^k$. Notice that for each t in the closed

interval $[a_i, b_i]$, the region $\sigma_i(t)$ embeds the $(k-1)$-dimensional rectangle s/s_i in the k-dimensional rectangle s by inserting value t at the index i corresponding to the missing dimension.

We define the boundary chain ∂s as follows:

$$\partial s = \sum_{i=1}^{k} (-1)^i (\sigma_i(a_i) - \sigma_i(b_i)). \tag{3}$$

In other words, each of the k indices i in s contributes two $(k-1)$-dimensional rectangles to the boundary chain, with their signs adjusted as shown. The sign adjustment provides the orientation. We embed the $(k-1)$-dimensional rectangles in s by replacing the closed interval $[a_i, b_i]$ at index i in s with each of its boundary values a_i and b_i.

We may integrate $k-1$ forms in $\mathbf{R}^k$ over this chain. For example, when $k = 1$ and $s = [a, b]$, s/s_1 is the rectangle of dimension zero containing the single point $Z = (\,)$, and $\sigma_1(t)(Z) = t$. We have

$$\partial s = (-1)(\sigma_1(a) - \sigma_1(b)) = \sigma_1(b) - \sigma_1(a).$$

When we integrate a zero form $f(x_1)$ over the chain (3), we get

$$\int_{\partial s} f = \int_{\sigma_1(b)} f - \int_{\sigma_1(a)} f = \int_Z f(\sigma_1(b)) - \int_Z f(\sigma_1(a))$$

$$= f(\sigma_1(b)(Z)) - f(\sigma_1(a)(Z)) = f(b) - f(a),$$

which agrees with what we said in § 1.

When $k = 2$ and $s = ([a_1, b_1], [a_2, b_2])$, we have

- $s/s_1 = [a_2, b_2]$ and $\sigma_1(t)(x_2) = (t, x_2)$ and $d\sigma_1(t)(x_2) = (0, dx_2)$.
- $s/s_2 = [a_1, b_1]$ and $\sigma_2(t)(x_1) = (x_1, t)$ and $d\sigma_2(t)(x_1) = (dx_1, 0)$.

Therefore

$$\partial s = -(\sigma_1(a_1)) - \sigma_1(b_1) + (\sigma_2(a_2) - \sigma_2(b_2)) = -\sigma_1(a_1) + \sigma_2(a_2) + \sigma_1(b_1) - \sigma_2(b_2).$$

When we integrate a one form $\omega = f_1\ dx_1 + f_2\ dx_2$ over the chain (3), we get

$$\int_{\partial s} \omega = -\int_{\sigma_1(a_1)} \omega + \int_{\sigma_2(a_2)} \omega + \int_{\sigma_1(b_1)} \omega + \int_{\sigma_2(b_2)} \omega.$$

By the definition of path integration for one forms (§ 1 (8) and (9)), this yields

$$= -\int_{s/s_1} \omega(\sigma_1(a_1)) \circ d\sigma_1(a_1) + \int_{s/s_2} \omega(\sigma_2(a_2)) \circ d\sigma_2(a_2) + \int_{s/s_1} \omega(\sigma_1(b_1)) \circ d\sigma_1(b_1) - \int_{s/s_2} \omega(\sigma_2(b_2)) \circ d\sigma_2(b_2)$$

$$= -\int_{a_2}^{b_2} f_2(a_1, x_2)\ dx_2 + \int_{a_1}^{b_1} f_1(x_1, a_2)\ dx_1 + \int_{a_2}^{b_2} f_2(b_1, x_2)\ dx_2 - \int_{a_1}^{b_1} f_1(x_1, b_2)\ dx_1.$$

If we represent the rectangle s in the Cartesian plane, then this integration traverses the boundary of the rectangle in a counterclockwise direction: for example, starting at the upper left hand corner (a_1, b_2) and moving down to (a_1, a_2), then right to (b_1, a_2), then up to (b_1, b_2), then left to (a_1, b_2).

Now we define the boundary chain for a k-dimensional region in $\mathbf{R}^n$. Let s be a k-dimensional rectangle, and let $\sigma\colon s \to \mathbf{R}^n$ be a k-dimensional region. For each i in $[1, k]$, the maps $\sigma_i(a_i)$ and $\sigma_i(b_i)$ embed s/s_i in s. Therefore we may compose σ with each $\sigma_i(a_i)$ and $\sigma_i(b_i)$; the composed maps are the $(k-1)$ dimensional regions $\sigma \circ \sigma_i(a_i)\colon s/s_i \to \mathbf{R}^n$ and $\sigma \circ \sigma_i(b_i)\colon s/s_i \to \mathbf{R}^n$. We may use these $(k-1)$-dimensional regions to define the oriented boundary chain for σ as follows:

$$\partial\sigma = \sum_{i=1}^{k} (-1)^i (\sigma \circ \sigma_i(a_i) - \sigma \circ \sigma_i(b_i)). \tag{4}$$

We may integrate $k-1$ forms in $\mathbf{R}^n$ over this chain.

Stokes' theorem for rectangles: The general statement of Stokes' theorem for a $k-1$ form ω in $\mathbf{R}^k$ and a k-dimensional rectangle s is

$$\int_s d\omega = \int_{\partial s} \omega. \tag{5}$$

This statement generalizes the corresponding statement that we made for closed intervals in § 1.

Proof: By the definition of a $k-1$ form in $\mathbf{R}^k$, we have $\omega = \sum_{j=1}^{k} f_j\ e_j$, where

$$e_j = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_k.$$

Because integration distributes over sums, it suffices to show the result when $\omega = f\ e_j$ for any j.

First we study the boundary integral on the right-hand side of (5). For each i in $[1, k]$ in the chain (3), we have $d\sigma_i(t)(x) = (dx_1, \ldots, dx_{i-1}, 0, dx_{i+1}, \ldots, dx_k)$, where we have used $x = (x_1, \ldots, x_{j-1}, x_{j+1,}, \ldots, x_k)$ as the $k-1$ coordinates of s/s_i. Therefore $\sigma_i(t)^*(f\ e_j)$ is $f(\sigma(t))\ e_j$ when $i = j$ and zero otherwise. Accordingly, when we integrate over the chain (3), all the terms except the j term drop out, and we have

$$\int_{\partial s} \omega = (-1)^j \int_{s/s_j} \Big[f(\sigma_j(a_j)) - f(\sigma_j(b_j)) \Big]\ e_j. \tag{6}$$

The right-hand integration makes sense because $f(\sigma_j(t))\ e_j$ is a simple $k-1$ form in the coordinates $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k$, so we can integrate it over the rectangle s/s_j.

Now we examine the integral on the left-hand side of (5). By the definition of the exterior derivative, we have

$$d\omega = (\sum_{i=1}^{k} D_i f\ dx_i) \wedge e_j.$$

All the dx_i terms except dx_j are present in e_j, so all the terms in the sum drop out except $D_j f\ dx_j$. Therefore

$$d\omega = D_j f\ dx_j \wedge e_j.$$

By carrying out $j-1$ transpositions, we can move dx_j into the missing place in e_j, yielding

$$d\omega = (-1)^{j-1} D_j f\ dx_1 \wedge \cdots \wedge dx_k.$$

The theory of integration (see, e.g., [Lang 1997]) tells us that when carrying out a multiple integration, we can perform the individual integrals in any order. Therefore we have

$$\int_s d\omega = (-1)^{j-1} \int_{s/s_j} \left[\int_{s_j} D_j f\ dx_j \right] dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \cdots dx_k = (-1)^{j-1} \int_{s/s_j} \left[\int_{s_j} D_j f\ dx_j \right] e_j.$$

By the fundamental theorem of calculus, this gives

$$\int_s d\omega = (-1)^{j-1} \int_{s/s_j} \Big[f(\sigma_j(b_j)) - f(\sigma_j(a_j)) \Big]\ e_j. \tag{7}$$

Putting (6) together with (7) yields the result. □

Stokes' theorem for regions: The general statement of Stokes' theorem for a $k-1$ form ω in $\mathbf{R}^n$ and a k-dimensional region $\sigma\colon s \to \mathbf{R}^n$ is

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega. \tag{8}$$

This statement generalizes the corresponding statement that we made for paths in § 1.

Proof: First we prove

$$d(\sigma^* \omega) = \sigma^*(d\omega). \tag{9}$$

If ω is a zero form f, then (9) just restates the chain rule. Otherwise, by linearity, it suffices to prove (9) when $\omega = f\ e$ for a basis vector e. On the right side, by the definition of the exterior derivative, we have $d\omega = df \wedge e$. By the definition of the pullback σ *, we have $(\sigma^*(d\omega))(y) = (df(\sigma(y)) \circ d\sigma(y)) \wedge \sigma^*(e)(y)$. By the chain rule, this is $d(f \circ \sigma)(y) \wedge \sigma^*(e)(y)$. On the left side, by the definition of the pullback, $\sigma^*\omega = (f \circ \sigma)(y)\ \sigma^*(e)(y)$. And by the definition of the exterior derivative, we have $d(\sigma^*\omega) = d(f \circ \sigma)(y) \wedge \sigma(e)(y)$. This proves (9).

Turning back to (8), using the definition of region integration and (9), we have

$$\int_\sigma d\omega = \int_s \sigma^*(d\omega) = \int_s d(\sigma^*\omega).$$

Then Stokes' theorem for rectangles gives

$$\int_\sigma d\omega = \int_{\partial s} \sigma^*\omega. \tag{10}$$

Comparing (10) with (8) and (3) with (4), we see it suffices to prove that for all i and for all t in $[a_i, b_i]$,

$$\int_{\sigma_i(t)} \sigma^*\omega = \int_{\sigma \circ \sigma_i(t)} \omega. \tag{11}$$

The integration on both sides of (11) is over the same rectangle s/s_i, so (11) holds if and only if we have

$$(\sigma_i(t)^* \circ \sigma^*)(\omega)(y) = (\sigma \circ \sigma_i(t))^*(\omega)(y). \tag{12}$$

at all y. For a zero form $\omega(x) = f(x)$, (12) is true, because both sides evaluate to $f(\sigma(\sigma_i(t)(y)))$. Otherwise, by linearity and because the pullback distributes over the wedge product of forms, it suffices to prove (12) for a primitive one form $\omega(x) = \lambda(x)$. Then by the definition of the pullback, the right-hand side of (12) is

$$\lambda((\sigma \circ \sigma_i(t))(y)) \circ d(\sigma \circ \sigma_i(t))(y).$$

By the chain rule, this is

$$\lambda((\sigma \circ \sigma_i(t))(y)) \circ d\sigma(\sigma_i(t)(y)) \circ d\sigma_i(t)(y). \tag{13}$$

Meanwhile, on the left-hand side of (12), we have $(\sigma^*\lambda)(y) = \lambda(\sigma(y)) \circ d\sigma(y)$. Applying $\sigma_i(t)^*$ to this last expression also yields (13). This establishes (12) and completes the proof. □

4. General Vector Spaces

Up to this point we have studied mappings $f: X \to Y$ where $X = \mathbf{R}^n$ and $Y = \mathbf{R}$. We now briefly consider more general vector spaces X and Y.

Generialized regions: We have studied the integration of regions $\sigma: s \to \mathbf{R}^n$, where s is a rectangle in $\mathbf{R}^k$. We carried out this integration according to the formula

$$\int_\sigma \omega = \int_s \sigma^*\omega, \tag{1}$$

where ω is an element of $\Omega^k(\mathbf{R}^n)$, i.e., a mapping $\mathbf{R}^n \to \bigwedge_{i=1}^{k} L(\mathbf{R}^n, \mathbf{R})$. We can easily generalize $\Omega^k(\mathbf{R}^n)$ to $\Omega^k(V)$, where V is a normed vector space of dimension n. An element ω of this space is a mapping $V \to \bigwedge_{i=1}^{k} L(V, \mathbf{R})$; we may represent such an element as a sum of primitive k forms, just as we did for $\Omega^k(\mathbf{R}^n)$. Further, to carry out the right-hand integral, we just need to integrate the simple k form $\sigma^*\omega$ in $\Omega^k(\mathbf{R}^k)$ over the rectangle s; and nothing in the definition of the pullback $\sigma^*\omega$ depends on the coordinates in $\mathbf{R}^n$. Therefore we may replace $\mathbf{R}^n$ with V in the integration: that is, we may use the formula (1) to integrate ω in $\Omega^k(V)$ over the region $\sigma: s \to V$. All of the previous results for region integration, e.g., Stokes' theorem, go through identically in this case.

One can generalize regions even further. Instead of integrating over rectangles that are mapped into regions, one can integrate over objects called **manifolds**. A real manifold locally "looks like" $\mathbf{R}^n$ (in a way that is made mathematically precise) but may have a different overall shape. For example, it may be shaped like a donut. Manifolds are studied in a branch of mathematics called differential geometry. They also have close connections to other areas of mathematics such as algebraic topology. The subject of **Riemann surfaces**, or complex manifolds of dimension

one, is a kind of nexus point within mathematics, sitting in the intersection of complex analysis, differential geometry, algebraic topology, and algebraic geometry.

Generalizing the vector space Y: The method of integration learned in first-year calculus is called **Riemann integration**, after the nineteenth-century mathematician Bernhard Riemann. The idea of Riemann integration is that if we wish to integrate a function $f(x)\colon \mathbf{R} \to \mathbf{R}$ over the closed interval $s = [a, b]$, then we construct a sequence of partitions $P_i(s) = \{\ s_{ij}\ \}$ of the interval s into smaller and smaller sub-intervals $s_{ij} = [a_{ij}, b_{ij}]$, and we define

$$\int_s f(x)\ dx = \lim_{i \to \infty} \sum_{[a_{ij},b_{ij}] \in P_i(s)} f(a_{ij})(b_{ij} - a_{ij}).$$

That is, the Riemann integral is the limit of the sums of the rectangles lying under $f(x)$ whose bases are the partition intervals.

Nothing in this definition requires $Y = \mathbf{R}$ in the map $f\colon \mathbf{R} \to Y$. All we need is to take limits of sums of terms of the form $f(a_{ij})(b_{ij} - a_{ij})$, and we can do that if f takes values in any normed vector space Y. Then we can define a space of k forms, where each form ω is a map $\omega\colon \mathbf{R}^n \to \bigwedge_{i=1}^{k} L(\mathbf{R}^n, Y)$.

For example, let $Y = \mathbf{R}^2$. We can represent each k form ω via the coordinate maps $\omega = (\omega_1, \omega_2)$. Then the theory of integration (see, e.g., [Lang 1997]) tells us that we can integrate coordinate by coordinate, i.e., we have

$$\int \omega = \int (\omega_1, \omega_2) = (\int \omega_1, \int \omega_2).$$

To take a concrete example:

$$\int (x\ dx, x^2\ dx) = (\frac{x^2}{2}, \frac{x^3}{3}).$$

Generalized integration: We can generalize integration even further, by getting completely away from intervals and rectangles in $\mathbf{R}^n$. We can define a **measure** μ in a general normed vector space V. The measure maps some of the subsets of V (called the measurable sets s) to nonnegative real numbers $\mu(s)$ (called the measure of s). This idea generalizes what we do for Riemann integration, where the closed intervals $s_{ij} = [a_{ij}, b_{ij}]$ are measurable sets with measure $\mu(s_{ij}) = b_{ij} - a_{ij}$.

Let W be a normed vector space and $f\colon V \to W$ be a map. To integrate f over a set s in V, we construct partitions $P_i(s)$ of s into measurable subsets s_{ij}. This generalizes what we did for Riemann integration, where the measurable sets were the closed intervals $[a_{ij}, b_{ij}]$. Then we define a sequence of **step maps** $f_i\colon s \to W$ such that at each i, and for each s_{ij}, f is defined and attains the same value $f_i(x) = w_{ij}$ at every point x of s_{ij}, except possibly for a set of points of measure zero. In one dimension, the picture is similar to the one for Riemann integration, except that instead of a smooth map with steps underneath it, the map itself provides the steps.

We construct the vector space of maps from V to W, and we put a norm on that space so we can take limits in it. We construct the sequence f_i of step maps so that

$$\lim_{i \to \infty} f_i = f$$

in that space, with that norm.

For each step map f_i, we define its integral analogously to what we did for Riemann integration:

$$\int_s f_i\ d\mu = \sum_{s_{ij} \in P_i(s)} w_{ij}\mu(s_{ij}).$$

That is, we sum the value of the map times the measure of the base set over all the base sets in the partition. Then we define

$$\int_s f\ d\mu = \lim_{i \to \infty} \int_s f_i\ d\mu.$$

This idea generalizes Riemann integration in a powerful way. The measure μ that is often used is called **Lebesgue measure**. You can learn more about this kind of integration theory in any book on graduate real analysis, e.g., [Lang 1993].

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