The Homology of Regions in $\mathbf{R}^n$

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This paper provides a brief introduction to the mathematical concept of **homology**. The motivating idea of homology is to characterize how shapes of various dimensions enclose holes in a topological space.

In this paper we focus on the homology of rectangular regions (i.e., paths and their higher-dimensional counterparts) in subsets of $\mathbf{R}^n$. There is a close connection between this kind of homology and the calculus of integration.[1]

This paper can provide background and motivation for more advanced discussions of homology, e.g., [Massey 1991]. These discussions tend to be abstract and technical, and it helps to motivate them with simple, concrete examples. Even if you are already familiar with homology theory, reading this paper may provide insight into why the "cycles" and "boundaries" of homology are given those names.

The prerequisites for reading this paper are a basic knowledge of sets, mappings, and point-set topology in $\mathbf{R}^n$, together with a basic knowledge of groups. See, e.g., §§ 1–5 of my paper *Definitions for Commutative Algebra*.

1. Rectangles, Regions, and Chains

To begin, we define the following basic concepts: k-dimensional rectangles, regions, and chains of regions.

Rectangles: A k-dimensional **rectangle** s is a subset of $\mathbf{R}^k$ formed by taking the Cartesian product of k closed intervals $s_i = [a_i, b_i]$, one for each dimension $i \in [1, k]$. In this paper we will assume that $b_i > a_i$ at each dimension i. We denote a rectangle as a list of closed intervals: $s = (s_1, \ldots, s_k) = ([a_1, b_1], \ldots, [a_k, b_k])$.

A zero-dimensional rectangle is an empty list ( ). A one-dimensional rectangle is a closed interval $[a, b]$. A two-dimensional rectangle is what we ordinarily think of as a rectangle, i.e., a set of points whose coordinates in the first dimension lie in the range $[a_1, b_1]$ and whose coordinates in the second dimension lie in the range $[a_2, b_2]$. In higher-dimensions, a k-dimensional rectangle is a k-dimensional rectangular solid.

Regions: A k-dimensional **region** σ is a continuous mapping from a k-dimensional rectangle s to $\mathbf{R}^n$. In the context of integration, we usually require that the mapping $\sigma\colon s \to \mathbf{R}^n$ be differentiable. Here we require only that the mapping σ be continuous.[2]

A zero-dimensional region is called a **point**. It maps the zero-dimensional rectangle ( ) to a point in $\mathbf{R}^n$. A one-dimensional region is called a **path**. It maps a closed interval $[a, b]$ to a one-dimensional set of points in $\mathbf{R}^n$. We call $\sigma(a)$ the **initial point** of the path σ, and we call $\sigma(b)$ the **terminal point** of σ. We call the pair $(\sigma(a), \sigma(b))$ the **endpoints** of σ.

A path σ is **closed** if its endpoints are equal, i.e., $\sigma(a) = \sigma(b)$. For example, the path $\sigma\colon [0, 2\pi] \to \mathbf{R}^2$ given by $x \mapsto (\cos x, \sin x)$ is a closed path. It starts at $(1, 0)$ and travels once counterclockwise around the unit circle centered at the origin. Its endpoints are $((1, 0), (1, 0))$.

The **reverse** of a k-dimensional region σ, which we will write σ^R, is the k-dimensional region defined by

$$\sigma^R(x_1, \ldots, x_k) = \sigma(a_1 + b_1 - x_1, \ldots, a_k + b_k - x_k).$$

It reverses the direction of travel along the image of σ in each of the k dimensions, starting at b_i and ending at a_i. For example, the reverse of the path $\sigma\colon [0, 2\pi] \to \mathbf{R}^2$ given by $x \mapsto (\cos x, \sin x)$ is the path σ^R given by $x \mapsto (\cos(2\pi - x), \sin(2\pi - x))$. It winds once around the unit circle clockwise.

Chains: A k-dimensional **chain** γ is a finite formal sum of terms t, where t is σ or $-\sigma$, and σ is a k-dimensional region. We call $-\sigma$ the **inverse** of the region σ. We write 0 to denote the empty sum (i.e., the unique sum with no terms). As usual, we write mt as a shorthand for $t + \cdots + t$ (m times), where $m \geq 0$ is a natural number. In particular,

[1] For more on this connection, see my papers *Integration in Real Vector Spaces* and *Calculus Over the Complex Numbers*.

[2] In [Massey 1991], it is assumed that each interval $[a_i, b_i]$ is $[0, 1]$, and a k-dimensional region is called a **singular k-cube**.

0σ and $0(-\sigma)$ both denote 0. We also write $(-m)\sigma$ for $m(-\sigma)$, and we write $m_1\sigma_1 - m_2\sigma_2$ for $m_1\sigma_1 + (-m_2)\sigma_2$. With this notation, we may write any chain in the form

$$\sum_{i=1}^{m} m_i\sigma_i, \tag{1}$$

where the m_i are nonzero integers, and $\sigma_i \neq \sigma_j$ for $i \neq j$.

We consider two chains to be equivalent if they may be put into a common form by reordering terms and by canceling pairs of terms σ and $-\sigma$. Under this equivalence, each chain has a unique representation in the form (1), up to the ordering of the terms in the sum. For example, $-\sigma_2 + \sigma_1 - \sigma_2$ is equivalent to $\sigma_1 - 2\sigma_2$, and $\sigma_1 - \sigma_1$ is equivalent to 0. We say that each of the σ_i in the canonical form is a **component region** of the chain γ.

For formal completeness, we allow k to range over the integers in the definition of a k-dimensional chain. That is, we allow -1-dimensional chains, -2-dimensional chains, etc. Since there are no regions of dimension less than zero, all k-dimensional chains for $k < 0$ are equal to zero.

The **reverse** of a chain γ, written γ^R, is formed by reversing each of its component regions, i.e.,

$$(\sum_{i=1}^{m} m_i\sigma_i)^R = \sum_{i=1}^{m} m_i\sigma_i^R.$$

In later sections we will see that the chains $-\sigma$ and σ^R are **homologous**, i.e., equivalent in the sense of homology. Therefore, we can think of changing the sign of σ as reversing the direction or orientation of σ.

There is an obvious addition rule for k-dimensional chains, i.e.,

$$(\sum_i m_i\sigma_i) + (\sum_i n_i\tau_i) = \sum_i m_i\sigma_i + \sum_i n_i\tau_i.$$

The plus sign on the left represents a sum of chains, and the plus sign on the right is the formal sum of regions in the definition of a chain. This rule makes the set of all k-dimensional chains into an additive group.

Nondegenerate regions and chains: Let $\sigma\colon s \to \mathbf{R}^n$ be a k-dimensional region. For each i in $[1, k]$, let $s\backslash s_i$ be the $k-1$-dimensional rectangle $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k)$ formed by deleting s_i from the list of intervals in s. For each t in $[a_i, b_i]$, let $\sigma_i(t)\colon s\backslash s_i \to \mathbf{R}^n$ be the $k-1$-dimensional region that takes each point $(x_1, \ldots, x_{k-1})$ in $s\backslash s_i$ to the point $(x_1, \ldots, x_{i-1}, t, x_i, \ldots, x_{k-1})$ in $\mathbf{R}^n$. Notice that $\sigma_i(t)$ inserts the constant value t for the deleted coordinate x_i.

We say that σ is **degenerate** if, for some i in $[1, k]$ and all pairs (t, u) of numbers in s_i, $\sigma \circ \sigma_i(t) = \sigma \circ \sigma_i(u)$. In this case, the value $\sigma(x)$ does not depend on the coordinate x_i. For example, the two-dimensional region σ given by $\sigma(x_1, x_2) \mapsto (x_1, x_1)$ is degenerate: it maps the two-dimensional set s to the one-dimensional set of points (x_1, x_1) such that x_1 lies in $[a_1, b_1]$.

If a k-dimensional region is not degenerate, then we say that it is **nondegenerate**. We say that a k-dimensional chain is nondegenerate if it is a formal sum of nondegenerate regions and their inverses. It is clear that the sum of nondegenerate chains is again nondegenerate. Henceforth we will consider only nondegenerate regions and chains.

Note that (1) when $k < 1$, no k-dimensional chain is degenerate; and (2) when $k > n$, every k-dimensional chain is degenerate.

2. Boundaries and Cycles

In this section we define the boundary of a rectangle, region, or chain. We also define the concept of a cycle, which extends the concept of a closed path (§ 1).

2.1. Boundaries

The boundary of a rectangle: The **boundary** of a k-dimensional rectangle s, for $k \geq 0$, is a $k-1$-dimensional chain ∂s that describes the endpoints of an interval (when $k = 1$), the edges of a rectangle (when $k = 2$), or the faces of a rectangular solid (when $k > 2$). Every boundary has an **orientation**, which associates an orientation (expressed as a sign) with each component of the boundary.

We define

$$\partial s = \sum_{i=1}^{k} (-1)^i(\sigma_i(a_i) - \sigma_i(b_i)), \tag{2}$$

where for each i $\sigma_i(t)$: $s \backslash s_i \to \mathbf{R}^n$ is the region

$$\sigma(t)(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{i-1}, t, x_i, \ldots x_{k-1})$$

that we defined in § 1. In other words, the boundary of a rectangle sums the pairs of opposing faces (or edges, or endpoints) of the rectangle, with a sign adjustment. The sign adjustment provides the orientation.

Note that when s is the zero-dimensional rectangle ( ), ∂s is the -1-dimensional chain 0.

The boundary of a region: The **boundary** of a k-dimensional region σ: $s \to \mathbf{R}^n$, for $k \geq 0$, is a $k-1$-dimensional chain $\partial\sigma$. It is formed by composing each of the component regions in the boundary ∂s with σ:

$$\partial\sigma = \sum_{i=1}^{k} (-1)^i (\sigma \circ \sigma_i(a_i) - \sigma \circ \sigma_i(b_i)). \tag{3}$$

The composition is well-defined because the images of $\sigma_i(a_i)$ and $\sigma_i(b_i)$ lie in s.

Notice that when $k = 0$, $\partial\sigma = 0$.

The boundary of a chain: The **boundary** of a k-dimensional chain γ, for k an integer, is a $k-1$-dimensional chain $\partial\gamma$. It is formed by applying the boundary operator to each of the component regions of γ:

$$\partial\gamma = \partial(\sum_{i=1}^{m} m_i\sigma_i) = \sum_{i=1}^{m} m_i \partial\sigma_i. \tag{4}$$

Notice that when $k < 1$, $\partial\gamma = 0$.

It is clear from the definitions that the boundary of the reverse of a chain is the reverse of the boundary, i.e.,

$$\partial(\gamma^R) = (\partial\gamma)^R.$$

It is also clear from the definitions that if γ is nondegenerate, then $\partial\gamma$ is nondegenerate.

A rectangle is a region, via the identity map, and a region is a chain. Therefore every boundary is the boundary of a chain. We will say that a chain γ is a $k-1$**-dimensional boundary** if it is the boundary of a k-dimensional chain.

2.2. Cycles

A k-dimensional **cycle** in $\mathbf{R}^n$ is a k-dimensional chain γ such that $\partial\gamma = 0$. For example, in the case that $k = 1$ and $n = 2$:

1. A closed path σ: $[a, b] \to \mathbf{R}^2$ is a one-dimensional cycle. Indeed, its boundary $\partial\sigma$ is $\sigma(b) - \sigma(a) = 0$.
2. The boundary ∂s of a rectangle $s = ([a_1, b_1], [a_2, b_2])$ is a one-dimensional cycle. Indeed, from (2) we see that

   $$\partial s = -\sigma_1(a_1) + \sigma_1(b_1) + \sigma_2(a_2) - \sigma_2(b_2)$$

   and

   $$\begin{aligned}\partial\partial s &= -\partial\sigma_1(a_1) + \partial\sigma_1(b_1) + \partial\sigma_2(a_2) - \partial\sigma_2(b_2) \\ &= -((a_1, b_2) - (a_1, a_2)) + ((b_1, b_2) - (b_1, a_2)) + ((b_1, a_2) - (a_1, a_2)) - ((b_1, b_2) - (a_1, b_2)).\end{aligned}$$

   After distributing the signs, all the terms cancel out to zero in pairs.

Each of these cases corresponds to what we intuitively think of as a "cycle." The first case is a single path that starts and ends at the same point. In the second case, if we put the terms of $\partial\sigma$ in the order $-\sigma_1(a_1)$, $\sigma_2(a_2)$, $\sigma_1(b_1)$, $-\sigma_2(b_2)$, and we think of the negative sign as reversing the direction of travel, then the chain $\partial\sigma$ starts at a point and goes around counterclockwise to end at the same point. For example, if the starting point is (a_1, b_2), then $\partial\sigma$ goes down to (a_1, a_2), right to (b_1, a_2), up to (b_1, b_2), and left to (a_1, b_2).

In higher dimensions, we can think of a "cycle" as a chain that encloses a volume of space. For example, the boundary of a k-dimensional rectangle is a cycle (we will show this in § 4.1).

We will say that a formal sum of regions γ is **reduced** if it contains no uncancelled pairs σ and $-\sigma$. We will say that a k-dimensional cycle γ is **composite** if γ is equivalent to a reduced formal sum $\gamma_1 + \gamma_2$, where each of γ_1 and γ_2 is a cycle, and neither γ_1 nor γ_2 is equal to zero. If a cycle is not composite, then we say that it is **atomic**. For example:

1. The boundary of a rectangle is an atomic cycle.
2. The sum of the boundaries of two rectangles is a composite cycle; it is the sum of two atomic cycles.

Every cycle is equivalent to a sum of atomic cycles.

3. The Homology of One-Dimensional Cycles

Homology is the algebra of cycles modulo boundaries at each dimension k. That is, for each dimension k we are concerned with k-dimensional cycles, and we consider two k-dimensional cycles γ_1 and γ_2 to be equivalent if the chain $\gamma_1 - \gamma_2$ is a k-dimensional boundary. When two cycles are equivalent in this sense, we say that they are **homologous**. In particular, every k-dimensional boundary $\partial\gamma$ is homologous to the k-dimensional chain 0; thus we say that $\partial\gamma$ is **homologous to zero**.

In this section, we consider the homology of one-dimensional cycles (i.e., chains of paths) and one-dimensional boundaries (i.e., boundaries of two-dimensional chains). In § 4 we will consider the homology of higher-dimensional regions.

3.1. One-Dimensional Boundaries in $\mathbf{R}^n$

First we show that in $\mathbf{R}^n$, every boundary is a cycle.

Proposition: Let γ be a one-dimensional boundary in $\mathbf{R}^n$. Then γ is a one-dimensional cycle, i.e., $\partial\gamma = 0$.

Proof: By hypothesis, there exists a two-dimensional chain γ' such that $\gamma = \partial\gamma'$. We must show that $\partial\partial\gamma' = 0$. From the definition of a chain, it suffices to show that $\partial\partial\sigma = 0$ for every two-dimensional region σ in $\mathbf{R}^n$. In § 2.2 we showed that $\partial\partial s = 0$ for a two-dimensional rectangle s. Since the terms of $\partial\partial s$ cancel out in pairs, and each term of $\partial\partial\sigma$ is a term of σ composed with a term of $\partial\partial s$, the terms of $\partial\partial\sigma$ cancel out in pairs as well. □

3.2. One-Dimensional Cycles in $\mathbf{R}^2$

Next we show that in $\mathbf{R}^2$, every cycle is a boundary. Together with the previous section, this result shows that the homology of one-dimensional cycles in $\mathbf{R}^2$ is trivial.

Proposition: Let γ be a one-dimensional cycle in $\mathbf{R}^2$. Then γ is a one-dimensional boundary, i.e., there exists a two-dimensional chain γ' such that $\partial\gamma' = \gamma$.

To prove this proposition, we will need some definitions and a lemma. Let γ be a one-dimensional atomic cycle in $\mathbf{R}^2$ (§ 2.2). Write γ in reduced form as $\sum_{i=1}^{m} c_i\sigma_i$, where each c_i is 1 or -1. Because the endpoints of the paths σ must cancel out in pairs when computing the boundary, one of the following cases must hold:

1. $m = 0$. In this case $\gamma = 0$, and the result is trivial.
2. $m = 1$, and γ consists of a single closed path.
3. $m = 2$, and γ consists of two distinct paths with the same endpoints, neither of which is closed. The two paths may have overlapping images; e.g., $\sigma + \sigma^R$ is allowed.
4. $m \geq 3$, no γ_i is closed, and each endpoint of σ_i coincides with an endpoint of a distinct path γ_j, $j \neq i$.

In all cases, each σ_i is a distinct path.

The **winding chain** $w(\gamma)$ is the chain that reverses the direction of all the paths appearing with a -1 coefficient in γ. That is,

$$w(\gamma) = \sum_{i=1}^{m} w(\sigma_i),$$

where $w(\sigma_i) = \sigma_i$ if $c_i = 1$ and $w(\sigma_i) = \sigma_i^R$ if $c_i = -1$. Pick some path $\tau_1 = w(\sigma_i)$ in $w(\gamma)$, and denote its initial point p. Then by the definition of a cycle there must be an ordering $\tau_1, \ldots, \tau_m = w(\sigma_{i_1}), \ldots, w(\sigma_{i_m})$ of the paths in $w(\gamma)$ such that traveling along each successive path in this order starts at p and ends at p.[3] Further, since γ is atomic, no subset of the paths in $w(\gamma)$ has this property, so we must traverse all the paths τ_i to get back to p, and the ordering is unique.

[3] At each step i, let p_i be the terminal point of τ_i. Either $p_i = p$, which is canceled by the initial point of τ_1, or there must be some path τ_{i+1} whose initial point cancels p_i.

Let $\tau_i: [a, b] \to \mathbf{R}^2$ be a path in $w(\gamma)$. Assume there is a point $t \in [a, b)$ such that τ intersects itself at $\tau(t)$, and the intersection does not extend on both sides of t. We call $\tau(t)$ an **intersection point** of the path τ_i with itself. For example, a point where τ_i makes an X shape is an intersection point, as is a point where it makes a Y shape; but a point where τ_i is coincident with itself on both sides is not an intersection point. If τ_i joins itself, is coincident, and diverges, then there are two intersection points, one where it joins itself and one where it diverges. Note that the domain $[a, b)$ in the definition is open at the terminal point, so that the terminal point of a closed path is not an intersection point.

Let τ_i and τ_j be paths in $w(\gamma)$. Let q be a point such that (a) q is in the image of τ_i and τ_j and (b) if the entire chain $w(\gamma)$ were merged into a single path, then q would be an intersection point of that path. We call such a point q an **intersection point** of the paths τ_i and τ_j . Note that we allow $i = j$ in this definition; in this case q is an intersection point of the path τ_i as defined above.

By definition, an atomic cycle γ may not have an intersection point at any endpoint of any path in $w(\gamma)$. However, $w(\gamma)$ may have intersection points at points of its paths away from the endpoints.

By continuity, each path in $w(\gamma)$ has either zero or finitely many intersection points. Indeed, if some path τ_i had infinitely many intersection points, then since there are finitely many paths in $w(\gamma)$ it would intersect some path τ_j in infinitely many points. Since the image of τ_i is compact, those points would have a limit point q. Since τ_j is is oscillating as the intersection points approach q, τ_j cannot converge to q; and since τ_j is continuous, q cannot be in the image of τ_j. But the image of the continuous map $\tau_j: [a, b] \to \mathbf{R}^2$ is closed and therefore contains its limit points. Since this is a contradiction, the number of intersection points must be finite.

We say that a one-dimensional cycle γ is **simple** if it is atomic and if none of the paths in $w(\gamma)$ has any intersection points. In this case $w(\gamma)$ is a loop in $\mathbf{R}^2$ with no overlap between its component paths except between the terminal point of each path and the initial point of the next path.

Let γ be an atomic one-dimensional cycle, and let q be an intersection point of τ_i and τ_j in $w(\gamma)$. A **subdivision** of γ with respect to the triple (q, τ_i, τ_j), is a pair of cycles (γ_1, γ_2) such that γ_1 starts at q, continues along τ_i to its terminal point, follows $w(\gamma)$ around to the initial point of τ_j, and continues to q; and γ_2 starts at q, continues along τ_j to its terminal point, follows $w(\gamma)$ around to the initial point of τ_i, and continues to q. This pair of cycles exists and is unique because γ is atomic. A **subdivision at** q is a subdivision with respect to any triple (q, τ_i, τ_j) such that q is an intersection point of τ_i and τ_j.

Let γ be any one-dimensional cycle. By applying subdivision repeatedly to each atomic cycle in the reduced form of γ, we may construct a cycle γ' such that (a) the winding chain of γ' traces out the same path as the winding chain of γ and (b) γ' is a sum of simple cycles. We call γ' a **partition** of γ into a sum of simple cycles. For example, let γ be a closed path that starts at a point p, loops around and back to p, then continues and loops on the other side and back to p in a figure-eight. Then γ is atomic but not simple, because it intersects itself at p. By subdividing at p, we can partition γ into a sum of two simple cycles, each of which is one of the loops of the figure-eight.

Lemma: Let γ be a one-dimensional cycle in $\mathbf{R}^2$, and let γ' be a partition of γ into simple cycles γ_i. If each γ_i is a boundary, then γ is a boundary.

Proof: By induction it suffices to show the result when we subdivide an atomic cycle into the cycle $\gamma_1 + \gamma_2$. By hypothesis there are two-dimensional regions $\sigma_1: s_1 \to \mathbf{R}^2$ and $\sigma_2: s_2 \to \mathbf{R}^2$ such that $\gamma_1 = \partial\sigma_1$ and $\gamma_2 = \partial\sigma_2$. By composing σ_1 and σ_2 with continuous transformations if necessary, we may assume that each σ_i maps one side of s_i to q. By scaling and translating the s_i, we can assume that the sides that map to q overlap and have opposite signs, so they cancel out. Let s be the rectangle formed by putting s_1 next to s_2 along the overlapping sides and deleting the overlapping sides. Let $\sigma: s \to \mathbf{R}^2$ be the region whose restriction to each σ_i is s_i. Then σ is a rectangle whose boundary is γ. □

Proof of the proposition: By the lemma, we may assume that γ is simple. Write $\gamma = \sum_{i=1}^{m} c_i\sigma_i$, where each c_i is 1 or -1. We proceed by induction on m.

The claim is obviously true in the case $m = 0$. Otherwise if $m \leq 3$ we can show the claim by direct construction:

1. If $m = 1$, then $\gamma = c\sigma$, where σ is a closed path. Let σ' be a two-dimensional region that in the x_1 dimension continuously shrinks σ towards its endpoint and in the x_2 dimension continuously chooses a point along the path chosen by x_1. We can let $\gamma' = c\sigma'$.

2. If $m = 2$, then $\gamma = c\sigma_2 - c\sigma_1$, where σ_1 and σ_2 have the same endpoints. We can let γ' be a two-dimensional region that in x_1 continuously transforms σ_1 into σ_2 and in x_2 continuously chooses a point along the path chosen by x_1.
3. If $e = 3$, then γ has three endpoints $\{p_1, p_2, p_3\}$. Let σ be the path that connects p_1 to p_2. We can let γ' be a two-dimensional region that in x_1 continuously transforms σ to p_3, conforming to the other two paths, and in x_2 continuously chooses a point along the path chosen in x_1.

Otherwise pick two paths σ_1 and σ_2 in C_1 with a mutual endpoint, and add a path σ_3 connecting the other two endpoints such that σ_3 together with the reverse paths σ_1^R and σ_2^R form a cycle C_3. $C_4 = C_1 + C_3$ is the sum of two boundaries (each a pair consisting of a path σ_i and its reverse) and a cycle C_5 with one less path than C_1. A sum of two boundaries is a boundary.[4] By the induction hypothesis C_5 is a boundary, so C_4 is a boundary. $-C_3$ is a boundary because it has three paths. Therefore $C_1 = C_4 + (-C_3)$ is a boundary. □

Theorem: The homology of one-dimensional cycles in $\mathbf{R}^2$ *is trivial.*

Proof: The result follows directly from the proposition and from § 3.1. □

3.3. One-Dimensional Cycles in $\mathbf{R}^2$ with a Hole

By the theorem in the previous section, the homology of one-dimensional cycles in $\mathbf{R}^2$ is trivial: every one-dimensional boundary in $\mathbf{R}^2$ is a cycle, and every one-dimensional cycle in $\mathbf{R}^2$ is a boundary. However, if we let T be the topological space consisting of $\mathbf{R}^2$ minus a point or "hole," then the homology of one-dimensional cycles in T is not trivial, because some cycles wind around the hole, and those cycles are not boundaries.

The winding number of a cycle around a point: Let γ be an atomic cycle (§ 2.2) with m terms σ or $-\sigma$ in its reduced form. Let p be any point not in the image of any of the component regions of γ. We wish to define an integer called the **winding number** of γ around p, denoted $W(\gamma, p)$, that will measure how many times the cycle γ winds around the point p.

As in § 3.2, let $w(\gamma)$ be the winding chain that results from replacing each term $-\sigma$ in γ with σ^R, and let $\tau_1, \ldots, \tau_m$ be the ordering of the paths in $w(\gamma)$ such that traveling along each successive path in this order starts at the initial point q of τ_1 and ends at q. Let $\tau\colon [0, 1] \to \mathbf{R}^2$ be a path that starts at q, follows the paths τ_i in order, and ends at q.[5] Represent points (x, y) in $\mathbf{R}^2$ in the standard way, with x increasing from left to right and y increasing from bottom to top. Let L be the line through p parallel to the x axis in $\mathbf{R}^2$. For any point $p' \neq p$ in $\mathbf{R}^2$, let L' be the line through p and p', and let the **canonical angle** $\theta_{p'}$ be the unique angle in the range $[0, 2\pi)$ that measures the counterclockwise displacement in radians from L to L'. Then there is a unique continuous function $\theta\colon [0, 1] \to \mathbf{R}$ such that $\theta(0) = \theta_{\tau(0)}$ and for all $t \in [0, 1]$, $\theta(t) - \theta_{\tau(t)}$ is an integer multiple of 2π.[6] Intuitively, $\theta(t)$ measures the canonical angle $\theta_{\tau(t)}$ of $\tau(t)$, except that $\theta(t)$ is not constrained to lie in $[0, 2\pi)$ and changes continuously, whereas $\theta_{\tau(t)}$ is constrained to lie in $[0, 2\pi)$ and is discontinuous when $\tau(t)$ crosses L.

We define the winding number as follows:

$$W(\gamma, p) = (1/2\pi)(\theta(1) - \theta(0)).$$

From the definition it is clear that θ is an integer that does not depend on the choice of the path τ, so the winding number is well-defined.

For example, let p be a point in $\mathbf{R}^2$, and let C_p be the unit circle centered on p. Then

- The winding number around p of a path that travels n times counterclockwise around C_p is n.
- The winding number around p of a path that travels n times clockwise around C_p is $-n$.

In general, a cycle γ is a sum of m atomic cycles γ_i. Thus we define

$$W(\gamma, p) = \sum_{i=1}^{m} W(\gamma_i, p).$$

For example, let γ be the sum of two closed paths, one of which travels n times counterclockwise around C_p and

[4] If $\partial\gamma_1' = \gamma_1$ and If $\partial\gamma_2' = \gamma_2$, then $\partial(\gamma_1' + \gamma_2') = \partial\gamma_1' + \partial\gamma_2' = \gamma_1 + \gamma_2$.

[5] For example, subdivide $[0, 1]$ into m equal intervals $I_i = [(i-1)/m, i/m]$, and for each t in I_i let $\tau'(t)$ be $\tau_i((mt - i + 1)(b_i - a_i))$, where $[a_i, b_i]$ is the domain of τ_i.

[6] For an explicit construction of this function using complex integration, see § 3.4 of my paper *Calculus Over the Complex Numbers*.

one of which travels n times clockwise around C_p. Then the winding number of γ around p is zero.

The following lemma lets us consider only simple cycles when computing the winding number:

Lemma 1: Let p be a point in $\mathbf{R}^2$, let γ be a one-dimensional cycle in $\mathbf{R}^2$, and let γ' be a partition of γ into simple cycles. Then $W(\gamma', p) = W(\gamma, p)$.

Proof: It suffices to show that if γ is atomic and we subdivide γ into $\gamma_1 + \gamma_2$, then $W(\gamma, p) = W(\gamma_1, p) + W(\gamma_2, p)$. But this is clear because we can wind around γ by starting at the intersection point of the subdivision, winding around γ_1, and winding around γ_2. □

The interior of a two-dimensional region: Fix a two-dimensional region $\sigma: s \to \mathbf{R}^2$. The **interior** of σ is the set of all points in the image of σ and not in the image of $\partial\sigma$.

Proposition 1: Let $\sigma: s \to \mathbf{R}^2$ be a two-dimensional region, and let p be a point of $\mathbf{R}^2$ not in the image of $\partial\sigma$. Then p is an element of the interior of σ if and only if $W(\partial\sigma, p) \neq 0$.

Proof: Partition $\partial\sigma$ into a sum of simple cycles γ_i according to Lemma 1. If $W(\partial\sigma, p) \neq 0$, then $W(\gamma_i, p) \neq 0$ for some i. Since γ_i is simple, either γ_i winds counterclockwise and encloses the image of σ on the left, or it winds clockwise and encloses the image of σ on the right. In either case, it winds around the image of σ.

Conversely, if p is in the set enclosed by γ_i, then because γ_i is simple γ_i winds around p with winding number 1 or -1. For every pair (i, j) such that $W(\gamma_i, p) = 1$ and $W(\gamma_j, p) = -1$, γ_i winds with p on the left, γ_j winds with p on the right, and the image of σ is between γ_i and γ_j, so p is not in the interior of σ. Therefore there must be some γ_i that winds around p with winding number 1, with no -1 to cancel it, or vice versa. □

If you are familiar with complex integration theory, then here is an alternate partial proof of the proposition, in the case where σ is a complex differentiable function, after identifying $\mathbf{R}^2$ with $\mathbf{C}$. From complex integration theory, we have

$$W(\partial\sigma, p) = \frac{1}{2\pi i} \int_{\partial\sigma} f \, dz,$$

where $f(z) = 1/(z - p)$. Suppose p is outside the interior of σ. By definition p it is outside the boundary of σ, so p is outside the image of σ, and f is holomorphic on the image of σ. In this case we may apply the generalized Stokes' theorem, yielding

$$W(\partial\sigma, p) = \frac{1}{2\pi i} \int_{\sigma} df \wedge dz = \frac{1}{2\pi i} \int_{\sigma} (f' \, dz) \wedge dz = 0.$$

Cycles that wind around a hole: Let T be the topological space formed by deleting a point p from $\mathbf{R}^2$ and retaining the topology of $\mathbf{R}^2$ on the remaining points. We call this topology the **induced topology** on $\mathbf{R} - \{p\}$. Let $\sigma: s \to \mathbf{R}^2$ be a region such that the image of $\partial\sigma$ does not contain p. The boundary $\partial\sigma$ is a cycle in $\mathbf{R}^2$, and it is also a cycle in T.

If the interior of σ does not contain p, then γ is a boundary in T. However, if the interior of σ does contain p, then γ is not a boundary in T. Since γ winds around p, if γ were a sum of boundaries of regions, then there would be some region $\sigma: s \to T$ whose boundary winds around p. By Proposition 1, the interior of σ would contain p. But this is impossible: since s would be compact and σ would be continuous, $\sigma(s)$ would be compact and therefore closed and bounded in $\mathbf{R}^2$. And p would be a limit point of $\sigma(s)$ in $\mathbf{R}^2$ that cannot be a member of $\sigma(s)$, so $\sigma(s)$ would not be closed in $\mathbf{R}^2$.

This argument shows that when we introduce a hole by deleting a point from $\mathbf{R}^2$, the homology of the resulting space is not trivial: there are cycles that are not boundaries, and they are exactly the cycles that wind around the hole.

Proposition 2: Let p be a point in $\mathbf{R}^2$, and let γ be a one-dimensional cycle in T. If $W(\gamma, p) \neq 0$, then γ is not a boundary.

Proof: If γ were a boundary, then we would have $\gamma = \partial\gamma'$, where γ' is a sum of two-dimensional regions σ_i. Since the winding number of $\partial\gamma$ is the sum of the winding numbers of $\partial\sigma_i$, some σ_i would wind around p, and its interior would contain p by Proposition 1. But this is impossible by the continuity argument that we made above. □

The winding number and homologous cycles: Again let $T = \mathbf{R}^2 - \{p\}$ with the induced topology.

Proposition 3: Let γ_1 and γ_2 be one-dimensional chains in T such that $W(\gamma_1, p) = W(\gamma_2, p)$. Then γ_1 and γ_2 are homologous, i.e., $\gamma_1 - \gamma_2$ is a one-dimensional boundary.

To prove the proposition we need two lemmas:

Lemma 2: Let γ_1 and γ_2 be simple one-dimensional cycles in T such that $W(\gamma_1, p) = 1$ and $W(\gamma_2, p) = -1$. Then $\gamma = \gamma_1 + \gamma_2$ is a boundary.

Proof: We will say that γ_1 is **inside** γ_2 if, for every ray R with endpoint p, for every point q_1 of intersection between R and the image of a path in γ_1, and for every point q_2 of intersection between R and the image of a path in γ_2, q_1 is at least as close to p along R as q_2.

If γ_1 is inside γ_2, possibly after interchanging the labels 1 and 2, then let σ be a path from one of the endpoints of a path in γ_1 to one of the endpoints of a path in γ_2. We can construct a two-dimensional region whose boundary is $\gamma = \gamma_1 + \sigma + \gamma_2 - \sigma$.

Otherwise we can represent γ as $\gamma_1' + \gamma_2 + \gamma_1''$, where (a) γ_1' is inside γ_2 and (b) $\gamma_1''$is the sum of all cycles formed by crossing γ_2 at a point q, following γ_1 until it crosses back over γ_2, and following γ_2 back to p. Then γ'' is a boundary because it does not wind around p, and $\gamma_1' + \gamma_2$ is a boundary by the argument in the previous paragraph. Hence γ is a sum of boundaries, so it is a boundary. □

Lemma 3: Let γ be a one-dimensional cycle in T. If $W(\gamma, p) = 0$, then γ is a k-dimensional boundary.

Proof: Partition γ into a sum of simple cycles γ_i (Lemma 1). Each γ_i that does not wind around p is a boundary. Match the remaining chains γ_i in pairs with winding number 1 and −1 and use Lemma 2. □

Proof of the proposition: Observe that $W(\gamma_1 - \gamma_2, p) = 0$, and use Lemma 3. □

From the proposition we can derive the following result:

Theorem: For any point p in $\mathbf{R}^2$, the homology of $T = \mathbf{R}^2 - \{p\}$ has the structure of $\mathbf{Z}$.

Proof: Proposition 3 says that two chains γ_1 and γ_2 in T are homologous if they have the same winding number with respect to p. If they have different winding numbers, then

$$W(\gamma_1 - \gamma_2, p) = W(\gamma_1, p) - W(\gamma_2, p) \neq 0.$$

In this case $\gamma_1 - \gamma_2$ is not a boundary by Proposition 2, and γ_1 is not homologous to γ_2. Therefore each winding number around p gives rise to a distinct homology class. Further, when we add two chains, the winding number of the sum is the sum of the winding numbers. □

For example:

1. let σ_1 be a path in T that winds once counterclockwise around p, and let σ_2 be a path in T that winds twice counterclockwise around p. Then the chain $\sigma_1 + \sigma_1$ is homologous to the chain σ_2.
2. Let σ be any path in T, and let σ^R be the path formed by reversing the direction of σ. By the definition of the winding number, we have $W(-\sigma, p) = W(\sigma^R, p)$. Therefore the chain $-\sigma$ is homologous to the chain σ^R. This is what we said in § 1.

3.4. One-Dimensional Cycles in $\mathbf{R}^n$

By projecting one-dimensional chains in $\mathbf{R}^n$ onto $\mathbf{R}^2$, one can extend the proposition in § 3.2 to $\mathbf{R}^n$. One can also show that the homology of one-dimensional cycles in $\mathbf{R}^n$ with a hole is trivial for $n \neq 2$. For $n > 2$, the intuition is that every cycle γ is the boundary of some two-dimensional chain γ'; and if γ' intersects a hole p in $\mathbf{R}^n$, then we can bend γ' slightly in a neighborhood of p to construct a chain γ'' that avoids p, without changing the boundary. We state these results without proof.

Theorem 1: The homology of one-dimensional cycles in $\mathbf{R}^n$ is trivial.

Theorem 2: Let p be a point in $\mathbf{R}^n$, and let $T = \mathbf{R}^n - \{p\}$. For $n \neq 2$, the homology of one-dimensional cycles in T is trivial.

4. Higher-Dimensional Regions

In this section we discuss the homology of higher-dimensional regions.

4.1. k-Dimensional Boundaries

First we prove the following generalization of the proposition in § 3.1:

Proposition: Let γ be a k-dimensional boundary in $\mathbf{R}^n$. Then $\partial\gamma$ is a k-dimensional cycle.

Proof: The result is trivial for $k < 1$. For $k \geq 1$, by hypothesis there exists a $k+1$-dimensional chain γ' such that $\gamma = \partial\gamma'$. We must show that $\partial\partial\gamma' = 0$. By the same argument that we made in § 3.1, it suffices to prove the result for a $k+1$-dimensional rectangle $s = (s_1, \ldots, s_{k+1})$. As in § 1, let $\sigma_i(t)\colon s/s_i \subseteq \mathbf{R}^k \to s \subseteq \mathbf{R}^{k+1}$ be the region that takes $(x_1, \ldots, x_k)$ to the point in with x_m at each coordinate $m < i$, t at coordinate i, and x_{m-1} at each coordinate $m > i$. From the definition of the boundary, we have

$$\partial s = \sum_{i=1}^{k+1} (-1)^i (\sigma_i(a_i) - \sigma_i(b_i)).$$

Let $\tau_j(t)\colon s/s_i/s_j \subseteq \mathbf{R}^{k-1} \to s/s_i \subseteq \mathbf{R}^k$ be the region that takes $(x_1, \ldots, x_{k-1})$ to the point with x_m at each coordinate $m < j$, t at coordinate j, and x_{m-1} at each coordinate $m > j$. Let a_{ij} denote a_j if $i > j$, otherwise $a_{i(j+1)}$, and similarly for b_{ij}. From the definition of the boundary again, we have

$$\partial\partial s = \sum_{i=1}^{k+1} (-1)^i \left[\sum_{j=1}^{k} (-1)^j (\sigma_i(a_i) \circ \tau_j(a_{ij}) - \sigma_i(a_i) \circ \tau_j(b_{ij})) - \sum_{j=1}^{k} (-1)^j (\sigma_i(b_i) \circ \tau_j(a_{ij}) - \sigma_i(b_i) \circ \tau_j(b_{ij})) \right]$$

$$= \sum_{S} (-1)^{i+j} \Big[\sigma_i(a_i) \circ \tau_j(a_{ij}) - \sigma_i(a_i) \circ \tau_j(b_{ij}) - \sigma_i(b_i) \circ \tau_j(a_{ij}) + \sigma_i(b_i) \circ \tau_j(b_{ij}) \Big], \tag{5}$$

where S is the set of all pairs (i, j) such that $i \in [1, k+1]$ and $j \in [1, k]$. Note that when $i \leq j$, the index j used in the inner sum is off by one from the index j used for the intervals s_j and b_j, so we write a_{ij} and b_{ij} instead of a_j and b_j.

For any $i, j \in [1, k+1]$, $i \neq j$, let s_{ij} be the pair of intervals (s_i, s_j), and let s/s_{ij} be the $k-1$-dimensional rectangle formed by deleting the intervals s_i and s_j from s. For any numbers $t \in s_i$ and $u \in s_j$, let $\sigma_{ij}(t, u)\colon s/s_{ij} \to s$ be the region that takes $(x_1, \ldots, x_{k-1})$ to the point with t at coordinate i, u at coordinate j, x_m at each coordinate $m < \min(i, j)$, x_{m-1} at each coordinate $\min(i, j) < m < \max(i, j)$, and x_{m-2} at each coordinate $m > \max(i, j)$. Let S_1 be the set of all pairs (i, j) in S such that $i > j$. Then it suffices to show that, for any two sets $\{t_i\}$ and $\{u_j\}$ of $k+1$ numbers,

$$\sum_{S} (-1)^{i+j} \sigma_i(t_i) \circ \tau_j(u_{ij}) = \sum_{S_1} (-1)^{i+j} \sigma_{ij}(t_i, u_j) - \sum_{S_1} (-1)^{i+j} \sigma_{ij}(u_i, t_j); \tag{6}$$

for if we substitute (6) for each term in (5), after replacing t_i with a_i or b_i and u_j with a_j or b_j as appropriate, then everything cancels out to zero.[7]

To show (6), let S_2 be the set of all pairs (i, j) in S such that $i \leq j$, and note that $S = S_1 \cup S_2$. Then

$$\sum_{S} (-1)^{i+j} \sigma_i(t_{ij}) \circ \tau_j(u_j) = \sum_{S_1} (-1)^{i+j} \sigma_i(t_{ij}) \circ \tau_j(u_{ij}) + \sum_{S_2} (-1)^{i+j} \sigma_i(t_{ij}) \circ \tau_j(u_{ij})$$

$$= \sum_{S_1} (-1)^{i+j} \sigma_i(t_i) \circ \tau_j(u_j) - \sum_{S_2} (-1)^{i+j+1} \sigma_i(t_i) \circ \tau_j(u_{j+1}). \tag{7}$$

In the first term on the right-hand side of (7), $\tau_j(u_j)$ places u_j at coordinate j of s/s_i, and then $\sigma_i(t_i)$ places t_i at coordinate $i > j$ of s. In the second placement, coordinate j of s/s_i becomes coordinate j of s. So the composition of regions is equivalent to $\sigma_{ij}(t_i, u_j)$, and the first term on the right-hand side of (7) equals the first term on the right-hand side of (6). In the second term on the right-hand side of (7), $\tau_j(u_{j+1})$ places u_{j+1} at coordinate j of s/s_i. Because $i \leq j$, when $\sigma_i(t_i)$ places t_i at coordinate i of s, the position of u_{j+1} moves up by one in passing from s/s_i to s. Therefore the second term on the right-hand side of (7) equals

[7] The first and last terms of (5) each expand to two terms that cancel each other, and the inner two terms of (5) expand to four terms that cancel each other in pairs.

$$-\sum_{S_2}(-1)^{i+(j+1)}\sigma_{i(j+1)}(t_i, u_{j+1}). \tag{8}$$

Now S_2 is the set of all (i, j) such that $i \in [1, k]$, $j \in [1, k]$, and $i \leq j$, so S_2 is the set of all (i, j) such that $j+1 \in [2, k+1]$, $i \in [1, k]$, and $j+1 > i$; whereas S_1 is the set of all (i, j) such that $i \in [2, k+1]$, $j \in [1, k]$, and $i > j$. Therefore we may replace i with j, replace $j+1$ with i, and replace S_2 with S_1 in (8), yielding

$$-\sum_{S_1}(-1)^{j+i}\sigma_{ji}(t_j, u_i). \tag{9}$$

Because $\sigma_{ji}(t_j, u_i) = \sigma_{ij}(u_i, t_j)$, (9) is equal to the second term on the right-hand side of (6). □

4.2. Homology Groups

To construct the homology of regions in a topological space $T \subseteq \mathbf{R}^n$, we make the following definitions for each integer k:

1. Let X_k be the set of nondegenerate k-dimensional chains in T. From § 1, we know that X_k has the structure of a commutative group.
2. Let $\partial_k \colon X_k \to X_{k-1}$ be the boundary map defined in § 2.1. From the definition it is clear that each ∂_k is a group homomorphism.
3. Let C_k be the set of k-dimensional cycles in X_k. By definition, C_k is the kernel of ∂_k.
4. Let B_k be the set of k-dimensional boundaries in X_k. By definition, B_k is the image of ∂_{k+1}.

For each k, by the proposition in § 4.1, we have $B_k \subseteq C_k$. Therefore we may form the quotient group $H_k = C_k/B_k$. As usual, H_k is the set of cosets

$$\{B_k + \gamma \mid \gamma \in C_k\}$$

with zero element B_k and addition rule

$$(B_k + \gamma) + (B_k + \gamma') = B_k + (\gamma + \gamma').$$

H_k is called the kth **homology group** associated with the topological space T. The homology group formalizes the concept of "cycles mod boundaries" that we introduced in § 3. In particular, two chains in C_k are homologous if and only if they are members of the same coset in H_k.

When we wish to be explicit about the space T, we write $H_k(T)$. For example, $H_1(\mathbf{R}^n)$ denotes the the group of one-dimensional cycles mod one-dimensional boundaries in $\mathbf{R}^n$.

Homology groups in $\mathbf{R}^n$: From the discussion in the previous sections, we know the following, for all $n > 0$:

1. $H_k(\mathbf{R}^n) = 0$ for all $k < 0$ or $k > n$. Indeed, in all these cases the group $X_k(\mathbf{R}^n)$ of k-dimensional chains is trivial.
2. $H_0(\mathbf{R}^n) = 0$. Indeed, any two points a and b are homologous in $\mathbf{R}^n$, because the chain $b - a$ is the boundary of a path from a to b.
3. $H_1(\mathbf{R}^n) = 0$. This is Theorem 1 from § 3.4.

It is possible to generalize Theorem 1 from § 3.4 and show that $H_k(\mathbf{R}^n) = 0$ for all k. The intuition here is that a k-dimensional cycle encloses a $k+1$-dimensional volume of space, so it is the boundary of a $k+1$-dimensional region.

Homology groups in $\mathbf{R}^n$ with a hole: Let 0 be the origin in $\mathbf{R}^n$, and let $T_n = \mathbf{R}^n - \{0\}$. In general, $H_k(T_n)$ is trivial except when $k = n - 1$.

1. When $n = 1$ and $k = 0$, there are three homology classes: one corresponding to all the positive points in $\mathbf{R}^1$, one corresponding to all the negative points in $\mathbf{R}^1$, and one corresponding to zero.
2. When $n = 2$ and $k = 1$, from § 3.3, we know that $H_1(T_2) = \mathbf{Z}$.
3. When $n > 2$, we can assign a higher-dimensional winding number $W(\gamma, 0)$ to each atomic cycle γ of dimension $n-1$ that does not intersect itself. $W(\gamma, 0)$ is 1 if γ encloses the origin with orientation 1, -1 if γ encloses the origin with orientation -1, and zero if γ does not enclose the origin. As in the $n = 2$ case, we say that the winding number of a sum of atomic cycles γ_i is the sum of the winding numbers of the γ_i. Using this construction, one can show that $H_{n-1}(T_n) = \mathbf{Z}$.

5. Homological Algebra

The homology of cycles and boundaries in vector spaces can be generalized to the study of purely algebraic objects. The algebra of these objects is called **homological algebra**. In this section we briefly discuss homological algebra.

5.1. Chain Complexes

Fix a topological space $T \subseteq \mathbf{R}^n$, and form the sequence of groups and maps

$$\cdots \overset{d_{k+1}}{\to} C_k \overset{d_k}{\to} C_{k-1} \overset{d_{k-1}}{\to} \cdots \tag{10}$$

for $k \in \mathbf{Z}$, where we have written d_k instead of ∂_k.[8] The sequence (10) is an example of a **chain complex**. In general, a chain complex is a sequence of R-modules C_k, for R a commutative ring, and maps d_k of the form (10) such that for all k we have $d_k \circ d_{k+1} = 0$, i.e., the image of d_{k+1} lies in the kernel of d_k. For the chain complex arising from cycles and boundaries in $\mathbf{R}^n$, the elements C_k are abelian groups, which are $\mathbf{Z}$-modules.

Fix a general chain complex. For each integer k the kth **homology module** H_k associated with the chain complex is the kernel of d_k modulo the image of d_{k+1}. We say that a chain complex is **exact at** k if $H_k = 0$, i.e., the image of d_{k+1} equals the kernel of d_k.

5.2. Cohomology

Cohomology is the study of **cochain complexes**. A cochain complex is a chain complex with the arrows reversed, i.e., a complex of the form

$$\cdots \overset{d_{k-1}}{\to} C_k \overset{d_k}{\to} C_{k+1} \overset{d_{k+1}}{\to} \cdots$$

where the C_k are R-modules, and for all k we have $d_k \circ d_{k-1} = 0$. For each integer k the kth **cohomology module** H_k associated with the cochain complex is the kernel of d_k modulo the image of d_{k-1}. We say that a cochain complex is **exact at** k if $H_k = 0$, i.e., the image of d_{k-1} equals the kernel of d_k.

For example, let C_k be the vector spaces of real differential forms of degree k, as discussed in *Integration in Vector Spaces*. Let d_k be the exterior derivative. Then these vector spaces and maps form a cochain complex. The associated cohomology is called **de Rham cohomology**. See, e.g., [Bott and Tu 1991].

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[8] Remember that $C_k = 0$ for all but finitely many k, so that the sequence (10) is actually a finite sequence surrounded on both sides by infinitely many zeros.