Holomorphic Maps Between Riemann Surfaces

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This paper develops the theory of holomorphic maps between Riemann surfaces. It assumes that you are familiar with the material covered in my papers *Calculus Over the Complex Numbers*, *Complex Charts on Topological Surfaces*, and *The Inverse and Implicit Mapping Theorems*.

1. General Properties

Recall that a Riemann surface $R = (T, A)$ is a topological surface on which we can do complex calculus. It is a topological space T equipped with a maximal atlas A consisting of charts $C_i = (U_i, \phi_i)$, where U_i is an open subset of T, and ϕ_i is a homeomorphism from U_i to an open subset of $\mathbf{C}$. We require that the charts be mutually compatible, in the sense that for any pair of charts (U_i, ϕ_i) and (U_j, ϕ_j), the transition function $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ is holomorphic on its domain of definition. If U_i and U_j are disjoint, then the domain of definition is empty, and ϕ_{ij} is trivially holomorphic.

Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces. Recall that a holomorphic map from R_1 to R_2 is a mapping $f : T_1 \to T_2$ of the topological spaces such that for each pair (C_i, C_j), where C_i is a chart of A_1 and C_j is a chart of A_2, the function $f_{ij} = \phi_j \circ f \circ \phi_i^{-1}$ is holomorphic on its domain of definition. If U_i and U_j are disjoint, then the domain of definition is empty, and f_{ij} is trivially holomorphic.

In this section we develop some general properties of holomorphic maps between Riemann surfaces. Most of these properties extend similar properties of holomorphic functions in the complex plane. In § 3, we will discuss additional properties specific to the case when the Riemann surface of the domain is compact.

1.1. The Inverse Image of a Point Under a Holomorphic Map

Connected open sets: Let T be a topological space, and let S be an open subset of T. Recall that S is **connected** if it cannot be expressed as the disjoint union of two nonempty open sets. For example:

- The open subset $(0, 1)$ of $\mathbf{R}$ is connected.
- The open subset $(-1, 0) \cup (0, 1)$ of $\mathbf{R}$ is not connected.

Note that we require the set S in the definition of a connected set to be open. Since a set must be open to be a union of two open sets, this definition doesn't make sense for sets that are not open (every such set is trivially "connected" in this sense, even, for example, a set of isolated points).

By definition the topological space T of a Riemann surface $R = (T, A)$ is open and connected.

Isolated points and discrete sets: Let S be a subset of a topological space T, and let s be a member of S. We say that s is an **isolated** point of S if there exists an open neighborhood of s in T that contains no point of S except s. We say that S is a **discrete** set if every point of S is isolated. For example:

- The set $S = \{2^{-n} \mid n > 0\} \subseteq \mathbf{R}$ is discrete.
- The set $S \cup \{0\}$ is not discrete, because the point 0 is not isolated.

Locally constant maps: Let T be a topological space, let S be a set, and let $f : T \to S$ be a map. Let p be a point in T. We say that f is **locally constant** at p if there is an open neighborhood U of p such that f is constant on U. We now state and prove a useful lemma about locally constant maps.

Recall that a map $f : T_1 \to T_2$ of topological spaces is **continuous** if, for every open set $S \subseteq T_2$, $f^{-1}(S)$ is open in T_1, where $f^{-1}(S)$ is the set of all points a in T_1 such that $f(a) = b$ for some b in T_2. Recall that a topological space T is **Hausdorff** if, for any two points a and b in T, there exist open subsets A and B of T such that $a \in A$, $b \in B$, and $A \cap B = \varnothing$. For any Riemann surface $R = (T, A)$, the space T is Hausdorff.

Lemma 1: Let T_1 and T_2 be topological spaces, and let $f: T_1 \to T_2$ be a continuous map. Assume that T_1 is connected and T_2 is Hausdorff. Let p be any point of T_1. If f is locally constant at p, then f is constant on T_1.

Proof: Assume that f is locally constant at p, and that $f(p) = q$. Let U be the set of elements u in T_1 such that f is locally constant at u with $f(u) = q$. Then U is nonempty, because it contains p; and it is open, because by assumption any element u has an open neighborhood U_u in U, so $U = \cup_{u \in U} U_u$. Let V be $T_1 - U$. Because $T_1 = U \cup V$, and U is open and nonempty, and $U \cap V = \varnothing$, and T_1 is connected, it will suffice to prove that V is open; for in this case we will have $V = T_1 - U = \varnothing$, i.e., $U = T_1$ as required.

If V is empty, then it is open. Otherwise let v be an element of V. Since T_2 is Hausdorff and $f(v) \neq q$, we can find an open neighborhood W_v of $f(v)$ such that $q \notin W_v$. Since f is continuous, $f^{-1}(W_v)$ is open. Further, $f^{-1}(W_v)$ contains v and is contained in V. Therefore $V = \cup_{v \in V} W_v$, so V is open. □

Maps in the complex plane: Let $U \subseteq \mathbf{C}$ be open, and let $f: U \to \mathbf{C}$ be a holomorphic map. Fix a point $q \in f(U)$. The theory of complex analysis tells us the following:

- *Property 1:* For each point $p \in f^{-1}(q)$, either (A) p is an isolated point of $f^{-1}(q)$, or (B) f is locally constant at p. This result holds because f has a convergent power series expansion at every point of $f^{-1}(q)$. See *Calculus over the Complex Numbers*, § 4.3.
- *Property 2:* If U is open and connected and f is nonconstant, then every point $p \in f^{-1}(q)$ is isolated, i.e., $f^{-1}(q)$ is discrete. This statement follows from Property 1 and from Lemma 1 above, treating U as a topological space via the subset topology.

For example:

1. Let U be the open set $(-1, 0) \cup (0, 1)$, which is not connected. Let $f: U \to \mathbf{C}$ be given by $f(z) = 1/2$ on $(-1, 0)$ and $f(z) = z$ on $(0, 1)$. Then the inverse image of $1/2$ is $f^{-1}(1/2) = (-1, 0) \cup \{1/2\}$. The point $1/2$ is isolated in $f^{-1}(1/2)$, so it satisfies Property 1(A) stated above. The points in $(0, 1)$ are not isolated in $f^{-1}(1/2)$, and they satisfy Property 1(B) stated above. $f^{-1}(1/2)$ is not discrete.
2. Let f be the function $z \mapsto z^2$ defined on $U = \mathbf{C}$. Then U is connected, and f is nonconstant. For each point $q \in \mathbf{C}$, the set $f^{-1}(q)$ is discrete. For example, $f^{-1}(1) = \{-1, 1\}$.

In real analysis, we can have a differentiable function that smoothly transitions from being constant to being nonconstant on a connected set. For example, the function given by $f(x) = 0$ for $x < 0$ and $f(x) = x^2$ for $x \geq 0$ is differentiable everywhere on the connected set $\mathbf{R}$ with $f'(0) = 0$. In complex analysis there are no such functions. Note that f has no second derivative, and a complex differentiable function must have derivatives of all orders.

Nonconstant maps between Riemann surfaces: For nonconstant holomorphic maps between Riemann surfaces, we have the following theorem:

Theorem: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f: R_1 \to R_2$ be a nonconstant holomorphic map. Fix a point $q \in f(T_1)$. Then $f^{-1}(q)$ is a discrete subset of T_1.

To prove the theorem we need a lemma:

Lemma 2: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f: R_1 \to R_2$ be a holomorphic map. Then the corresponding map $f: T_1 \to T_2$ of the topological spaces is continuous.

Proof: Let V be an open subset of T_2. We want to show that $f^{-1}(V)$ is an open subset of T_1. If $f^{-1}(V) = \varnothing$, then it is open. Otherwise choose an element a of $f^{-1}(V)$, and let $b = f(a)$. We need to find an open set X_a such that $a \in X_a$ and $X_a \subseteq f^{-1}(V)$. Then $f^{-1}(V)$ will be the union $\bigcup_{a \in f^{-1}(V)} X_a$ and will therefore be open.

Choose charts $C_1 = (U_1, \phi_1)$ containing a and $C_2 = (U_2, \phi_2)$ containing b, and let $F = \phi_2 \circ f \circ \phi_1^{-1}$. Let $W = V \cap U_2$. Then W is an open set of T_2 that contains b. $\phi_2(W)$ is open, and F is continuous, so $F^{-1}(\phi_2(W)) = \phi_1(f^{-1}(W) \cap U_1)$ is open. Let $X_a = \phi^{-1}(F^{-1}(\phi_2(W))) = f^{-1}(W) \cap U_1$. Because ϕ_1^{-1} is a homeomorphism, X_a is open. Further, X_a contains a and is contained in $f^{-1}(V)$. Therefore X_a is the required open neighborhood of a. □

Proof of the theorem: It will suffice to prove, for f, Property 1 stated above for maps in the complex plane. The result then follows from Lemmas 1 and 2.

To prove Property 1, fix a point $p \in T_1$, and let $q = f(p)$. Choose a chart $C_1 = (U_1, \phi_1)$ of A_1 and a chart $C_2 = (U_2, \phi_2)$ of A_2 such that $p \in U_1$ and $q \in U_2$. By composing with translation maps if necessary, we can ensure that U_1 is centered at p and U_2 is centered at q, so assume this is true, i.e., $\phi_1(p) = 0$ and $\phi_2(q) = 0$. Then $F = \phi_2 \circ f \circ \phi_1^{-1}$ is holomorphic, so Property 1 holds for F at $0 \in F^{-1}(0)$. Suppose Property 1(A) holds for F, i.e., 0 is an isolated point of $F^{-1}(0)$. Then there exists an open neighborhood V of 0 such that for no point $v \in V$ except 0 we have $F(v) = 0$. In this case $W = \phi_1^{-1}(V)$ is an open neighborhood of p such that for no point $w \in W$ except p we have $f(w) = q$. Therefore p is an isolated point of $f^{-1}(q)$, so Property 1(A) holds for f. Now suppose that Property 1(B) holds for F, i.e., there exists an open neighborhood V of 0 such that F is constant on V. Let $W = V \cap \phi(U_1)$. If $f(\phi_1^{-1}(W))$ contains more than one point, then ϕ_2 cannot be a homeomorphism, because it maps all such points to zero. Therefore f must be constant in the open neighborhood $\phi_1^{-1}(W)$ of p, and Property 1(B) holds for f. □

Because a homeomorphism is injective, we also have the following corollary:

Corollary: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f\colon R_1 \to R_2$ be a nonconstant holomorphic map. Fix a point p in T_1, and let $q = f(p)$. Choose charts $C_1 = (U_1, \phi_1)$ of A_1 containing p and $C_2 = (U_2, \phi_2)$ of A_2 containing q. Let F be the holomorphic function $\phi_2 \circ f \circ \phi_1^{-1}$. Then for any point a in $F^{-1}(\phi_2(q))$, there exists a connected open neighborhood U of a in which F is defined and nonconstant.

Proof: By the previous theorem, $f^{-1}(q)$ is a discrete subset of T_1. $b = \phi_1^{-1}(a)$ is a member of this set. Choose an open neighborhood V of b such that for no point v in v except b do we have $f(b) = q$. Let U be an open ball around a in $\phi_1(V)$. This is the required connected open neighborhood of a. □

This corollary will let us apply results for nonconstant holomorphic functions on connected open sets in **C** to holomorphic maps between Riemann surfaces.

1.2. The Inverse Mapping Theorem

Let f be a complex function that is defined and holomorphic on an open neighborhood U of a point a in **C**. From *Calculus Over the Complex Numbers*, we know that f has continuous derivatives of all orders at each point in U. Suppose further that $f'(a) \neq 0$. Then $Df(a)$ is invertible as a linear map. Therefore, by the inverse mapping theorem, there exists an open set $V \subseteq U$ containing a such that the function $g\colon V \to f(V)$ given by $g(z) = f(z)$ is bijective and has a differentiable inverse, i.e., is biholomorphic. When a function is biholomorphic, we also say that it is an **analytic isomorphism**. Therefore we say that g is a **local analytic isomorphism** for f on V.

For a proof of the inverse mapping theorem in the general case of a finite-dimensional vector space over **R** or **C**, see my paper *The Inverse and Implicit Mapping Theorems*. For an alternate proof that is specialized to the case of a holomorphic function expressed as a convergent power series, see [Lang 1999], II, Theorem 6.1.

Conversely, if f has a local analytic isomorphism g in a neighborhood V of a, then we must have $f'(a) \neq 0$. Indeed, $g^{-1} \circ g$ is the identity on V, so we must have $(g^{-1})'(g(a))g'(a) = 1$. Therefore $f'(a) = g'(a) \neq 0$.

1.3. The Multiplicity of a Holomorphic Mapping at a Point

Nonconstant functions on connected open sets in the complex plane: Suppose f is a complex function that is defined and holomorphic on a connected open neighborhood U of a point a in **C**. Suppose further that f is nonconstant on U. We claim that there exist a biholomorphic function G, defined in an open neighborhood V of a, and an integer $m > 0$ such that $f(z) = f(a) + G(z)^m$ on V. Notice that this statement extends the inverse mapping theorem for a nonconstant functions on a connected open set: when $f'(a) \neq 0$, $m = 1$, and $f(z) = f(a) + G(z)$ is biholomorphic in a neighborhood of a. When $f'(a) = 0$, $m > 1$. In this case, $G(z)$ is biholomorphic in a neighborhood of a, but $G(z)^m$ is not (it is holomorphic, but not locally invertible). The integer m is unique and is called the **multiplicity** of f at a.

To show that m and G exist, let $h = z - a$, and write

$$f(a + h) = f(a) + P(h),$$

where P is a power series expansion with $P(0) = 0$. Let $m > 0$ be the order of P, i.e., the index of the first nonzero term in P. This order must be finite, because U is open and connected, and f is nonconstant, so by § 1.1, a is an isolated point of $f^{-1}(a)$. Then we can write

$$f(a + h) = f(a) + h^m Q(h)$$

where $Q(0) \neq 0$. By continuity there exists an open neighborhood W of 0 such that $Q(h) \neq 0$ on W. When $z \neq 0$, $(z^m)' = mz^{m-1}$ is nonzero, so by the inverse mapping theorem $z^{1/m}$ exists. Therefore on W we can write

$$f(a+h) = f(a) + (hQ(h)^{1/m})^m.$$

Let $G(z) = hQ(h)^{1/m} = (z-a)Q(z-a)^{1/m}$. By the product rule,

$$G'(z) = Q(z-a)^{1/m} + (z-a)(Q(z-a)^{1/m})',$$

so $G'(a) = Q(0)^{1/m} \neq 0$. Therefore G is biholomorphic in some open neighborhood V of a, and $f(z) = f(a) + G(z)^m$ on V, as was to be shown.

To show that m is unique, suppose there exist integers $m_1, m_2 > 0$ and power series $P_1(h)$ and $P_2(h)$, each biholomorphic in a neighborhood of zero, such that

$$f(a+h) - f(a) = P_1(h)^{m_1} = P_2(h)^{m_2}$$

on the intersection V of the two neighborhoods. Then V is an open neighborhood of zero, and for each i we have

1. $P_i(0) = 0$, so ord $P_i > 0$.
2. P_i is biholomorphic on V, so $P_i'(0) \neq 0$. Therefore ord $P_i < 2$.

Statements (1) and (2) imply ord $P_i = 1$, i.e., $P_i = zQ_i$, with $Q_i(0) \neq 0$. Then $(zQ_1)^{m_1} = (zQ_2)^{m_2}$, and in a small neighborhood of zero we have $Q_1(z)^{m_1}/Q_2(z)^{m_2} = z^{m_2 - m_1}$. As z tends to zero, the norm of the left-hand side tends to a finite positive number, and the norm of the right-hand side either tends to zero or grows arbitrarily large, unless $m_1 = m_2$. Therefore we must have $m_1 = m_2$, as was to be shown.

As an example, consider the complex function $f(z) = z^2$, which is holomorphic on all of $\mathbf{C}$. At $z = 0$, $f'(0) = 0$, and f has multiplicity two. At a point $a \neq 0$, $f'(a) \neq 0$, and we have the Taylor series expansion

$$f(z) = a^2 + 2a(z-a) + (z-a)^2.$$

Therefore f has multiplicity one.

Finally, the proof shows that the multiplicity of f at a is equal to the order at a of the function $F(z) = f(z) - f(a)$. It is also equal to one plus the order of $f'(z)$ at a, because the formal derivative of a power series deletes the constant term and reduces the exponents of the remaining terms by one. See *Calculus Over the Complex Numbers*, § 4.3.

Nonconstant maps between Riemann surfaces: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f\colon R_1 \to R_2$ be a nonconstant holomorphic map. Fix a point a of T_1. Choose a chart $C_1 = (U_1, \phi_1)$ of A_1 centered at a and a chart $C_2 = (U_2, \phi_2)$ of A_2 centered at $f(a)$. By the definition of a holomorphic map between Riemann surfaces, the function $F = \phi_2 \circ f \circ \phi_1^{-1}$ is defined and holomorphic on $\phi_1(U_1) \subseteq \mathbf{C}$, and $F(0) = 0$.

By the corollary at the end of § 1.1, there exists a connected open neighborhood of zero in which F is defined and nonconstant. Therefore, by the results presented above for holomorphic maps on connected open sets in the complex plane, there exists a unique integer $m > 0$ such that for some open neighborhood V of zero and some biholomorphic function G on V, we have $F(z) = G(z)^m$ for all $z \in V$. Then

$$\begin{aligned}
z^m &= F \circ G^{-1} \\
&= (\phi_2 \circ f \circ \phi_1^{-1}) \circ G^{-1} \\
&= \phi_2 \circ f \circ (\phi_1^{-1} \circ G^{-1}) \\
&= \phi_2 \circ f \circ (G \circ \phi_1)^{-1}.
\end{aligned}$$

Let $\psi = G \circ \phi_1$. Then $C = (\psi^{-1}(V), \psi)$ is a chart of A_1 centered at a, and $\phi_2 \circ f \circ \psi = z^m$. Thus we have the following result:

Theorem: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces. Let $f\colon R_1 \to R_2$ be a nonconstant holomorphic map, and fix a point a of T_1. For a unique integer $m > 0$, we can choose a chart $C_1 = (U_1, \phi_1)$ of A_1 centered on a and a chart $C_2 = (U_2, \phi_2)$ of A_2 centered on $f(a)$ such that $F = \phi_2 \circ f \circ \phi_1^{-1} = z^m$.

Proof: We have already shown all but the uniqueness of m. To show uniqueness, suppose we have a chart $D_1 = (V_1, \psi_1)$ of A_1 centered on a and a chart $D_2 = (V_2, \psi_2)$ of A_2 centered on $f(a)$ and an integer n such that $\psi_2 \circ f \circ \psi_1^{-1} = z^n$. Then on $\phi_1(U_1) \cap \psi_1(D_1)$ we have

$$z^n = \psi_2 \circ (\phi_2^{-1} \circ \phi_2) \circ f \circ (\phi_1^{-1} \circ \phi_1) \circ \psi_1^{-1}$$

$$= (\psi_2 \circ \phi_2^{-1}) \circ F \circ (\phi_1 \circ \psi_1^{-1}).$$

Therefore

$$F(z) = z^m = (\phi_2 \circ \psi_2^{-1})([(\psi_1 \circ \phi_1^{-1})(z)]^n) = g(h(z)^n),$$

where g and h are biholomorphic and take zero to zero. Thus we have $g^{-1}(z^m) = h(z)^n$, where g^{-1} is biholomorphic and takes zero to zero. By the argument given above for complex functions, g^{-1} and h each have order one, so we can write

$$z^m P(z) = z^n Q(z),$$

where $P(0) \neq 0$ and $Q(0) \neq 0$ Then by the argument given for complex functions we have $m = n$. □

Again the integer m is called the **multiplicity** of the map f at the point a. The function $F(z) = z^m$ is called the **local normal form** of f at a.

Ramification points and branch points: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f\colon R_1 \to R_2$ be a holomorphic map. Let $m_f(p)$ denote the multiplicity of f at p.

- A point p of T_1 for which $m_f(p) > 1$ is called a **ramification point** of f.
- A point q of T_2 such that $f^{-1}(q)$ contains a ramification point of f is called a **branch point** of f.

For example, let $f = z^2 + 1$ be the holomorphic map from **C** to **C**, considering **C** as a Riemann surface. Then 0 is ramification point, and 1 is a branch point.[1]

1.4. The Open Mapping Theorem

Nonconstant functions on connected open sets in the complex plane: We now use the results from § 1.2 and § 1.3 to show that a nonconstant holomorphic function on a connected open set in the complex plane is an open mapping:

Theorem 1: Let $U \subseteq \mathbf{C}$ be a connected open set, and let $f\colon U \to \mathbf{C}$ be a nonconstant holomorphic function. Then f is an open mapping, i.e., for all open sets $V \subseteq U$, $f(V)$ is an open set.

To prove the theorem, we need a lemma:

Lemma: For all integers $n > 0$, the function $f(z) = z^n$ is an open mapping.

Proof: Let $U \subseteq \mathbf{C}$ be an open set, let b be a point of $f(U)$, and let a be a point of $f^{-1}(b)$. We need to show that there is an open set containing b and contained in $f(U)$. If $b = 0$, then $a = 0$. Because V is open, we may choose an open ball $B(0,r) \subseteq U$. Then $f(B(0,r)) = B(0,r^n)$, so $B(0,r^n) \subseteq f(U)$, and the requirement is satisfied in this case. If $b \neq 0$, then $a \neq 0$. In this case $f'(a) \neq 0$, and so by § 1.2, f is biholomorphic in an open neighborhood $V \subseteq U$. A biholomorphic map is open, so $f(V)$ is open, and $b \in f(V) \subseteq f(U)$. So the requirement is satisfied in this case as well. □

Proof of the theorem: Let $U \subseteq \mathbf{C}$ be an open set, let b be a point of $f(U)$, and let a be a point of $f^{-1}(b)$. By § 1.3, there is an open neighborhood $V \subseteq U$ of a on which $f(z) = f(a) + G(z)^m$, for $m > 0$ and G biholomorphic. Then on V, f is a composition of G, a power map, and a translation map, all of which are open. Therefore $f(V)$ is open, and $b \in f(V) \subseteq f(U)$. □

Nonconstant maps between Riemann surfaces: For Riemann surfaces, we have the following analogous result:

Theorem 2: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f\colon R_1 \to R_2$ be a nonconstant holomorphic map. Then f is an open mapping, i.e., for all open sets $V \subseteq T_1$, $f(V)$ is an open set in T_2.

Proof: Fix a point q in $f(V)$ and a point p in V such that $f(p) = q$. We want to show the existence of an open neighborhood of q contained in $f(V)$. Choose charts C_1 centered at p and C_2 centered at q, and let F be the holomorphic map between the charts. Per § 1.1, choose a connected open neighborhood around p in which f is defined and non-constant, and intersect that neighborhood with $V \cap U_i$. Call the resulting set W. By restricting to an open ball around $\phi_1(p)$ contained in $\phi_1(W)$ if necessary, we can assume that W and $\phi_1(W)$ are connected open sets. By the theorem for nonconstant functions on connected open sets in the complex plane, $F(\phi_1(W))$ is open in **C**. Then $f(W) = \phi_2^{-1}(F(\phi_1(W)))$ is open and is the required open neighborhood of q. □

[1] This use of the terms "ramification point" and "branch point" is standard but arbitrary, since ramification means branch in Latin. You just have to remember that ramification points lie in R_1 and branch points lie in R_2.

1.5. Meromorphic Functions

In the complex plane: Let $U \subseteq \mathbf{C}$ be an open set. Recall that a meromorphic function on U is a partial function f on U with the following properties:

1. f is defined and holomorphic on a set $U - P$, where P is a discrete set of poles.
2. At each point b of U, f has a Laurent series expansion
$$f(z) = \sum_{j=n}^{\infty} a_j (z-b)^j,$$
where n is an integer. For $b \notin P$, we have $n \geq 0$, and the Laurent series expansion is a power series expansion. For $b \in P$, we have $n < 0$.

At each point b of U, we define the **order** of f at b, written $\operatorname{ord}_b f$, as follows:

1. If f is identically zero in an open ball around b, then $\operatorname{ord}_b f = \infty$.
2. Otherwise $\operatorname{ord}_b f$ is the smallest index of any nonzero term in the Laurent expansion of f at b.

A finite order is one of $-n$, zero, or m, where n and m are positive integers. If the order is $-n$, then b lies in P, and f has a pole of order n at b. If the order is zero, then b lies in $U - P$, and $f(b) \neq 0$. If the order is m, then b lies in $U - P$, $f(b) = 0$, and f has a zero of order m at b.

For example, the function $f(z) = 1/z$ is a Laurent series expansion at zero. In this case $U = \mathbf{C}$ and $P = \{0\}$. The function f has a pole of order one (also called a simple pole) at zero. At every point $b \neq 0$ in $\mathbf{C}$, f is holomorphic and has the power series expansion given by the Taylor series
$$f(z) = \frac{1}{b} - \frac{(z-b)}{b^2} + \frac{(z-b)^2}{b^3} - \cdots = \frac{1}{b} \sum_{j=0}^{\infty} g(z)^j,$$
where $g(z) = -(z-b)/b$. By the convergence of the geometric series, this series converges to
$$(1/b)(1/(1-g(z))) = 1/z$$
for all z such that $|z-b| < |b|$. Thus the order of f is -1 at zero and zero at every other point of $\mathbf{C}$.

Recall also the following facts:

1. $\operatorname{ord}_b f = \operatorname{Res}_b f'/f$, where $\operatorname{Res}_b g$ denotes the **residue** of the meromorphic function g at b, i.e., the coefficient of the $1/z$ term in the Laurent series expansion of g at b. See *Calculus Over the Complex Numbers*, § 6.2. We may also write $\operatorname{ord}_b f = \operatorname{Res}_b df/f$, where df is the meromorphic one form $f'\,dz$. By the definition of the residue of a meromorphic one form, the two formulas are equivalent. See *Calculus Over the Complex Numbers*, § 6.3.
2. Let ω be a meromorphic one form (i.e., a one form $f\,dz$, where f is a meromorphic function) on an open set U. If $\phi: U \to \phi(U)$ is a biholomorphic function, then $\operatorname{Res}_b \omega = \operatorname{Res}_{\phi^{-1}(b)} \phi^* \omega$, where ϕ^* denotes the pullback with respect to ϕ. See *Calculus Over the Complex Numbers*, § 6.3.

On a Riemann surface: Let $R = (T, A)$ be a Riemann surface. A **meromorphic function** on R is partial function $f: T \to \mathbf{C}$, defined except on a discrete set of points of T (the poles of f), such that for each chart $C_i = (U_i, \phi_i)$ of A the local partial function $f_i = f \circ \phi_i^{-1}: \phi_i(U) \to \mathbf{C}$ is meromorphic.

Let $R = (T, A)$ be a Riemann surface, and let $f: R \to \mathbf{C}$ be a meromorphic function. Fix a point p in T, and choose a chart $C_i = (U_i, \phi_i)$ containing p. The **order** of f at p with respect to the chart C_i is the order at $\phi_i(p)$ of the meromorphic function $f \circ \phi_i^{-1}$. We now show that this order is independent of the choice of chart.

Let $C_1 = (U_1, \phi_1)$ and $C_2 = (U_2, \phi_2)$ be two charts containing p. For each i, let $f_i = f \circ \phi_i^{-1}$, and let $\omega_i = df_i/f_i$. Then from fact 1 stated above the order of f at p with respect to chart C_i is $\operatorname{Res}_{\phi_i(p)} \omega_i$. Let ϕ be the transition function $\phi_1 \circ \phi_2^{-1}$ from C_2 to C_1. We claim that $\phi^* \omega_1 = \omega_2$. Indeed, we have
$$\begin{aligned}(\phi^* \omega_1)(z) &= (df_1(\phi(z)) \circ d\phi(z))/f_1(\phi(z)) \\ &= d(f_1 \circ \phi)(z)/(f_1 \circ \phi)(z) \\ &= df_2(z)/f_2(z)\end{aligned}$$

$$= \omega_2(z).$$

Now from fact 2 stated above we know

$$\mathrm{Res}_{\phi_1(p)}\ \omega_1 = \mathrm{Res}_{\phi^{-1}(\phi_1(p))}\ \phi^* \omega_1 = \mathrm{Res}_{\phi_2(p)}\ \omega_2,$$

which was to be shown.[2]

Therefore the order of f at p is independent of the chart, and we can write $\mathrm{ord}_p\ f$, without specifying a chart.

The associated holomorphic map: Let f be a meromorphic function on a Riemann surface R, and let P be the poles of f. By the argument we made in § 5.4 of *Complex Charts on Topological Surfaces*, f has an associated holomorphic map g from R to the Riemann sphere $\mathbf{C}_\infty$. For $p \notin P$, this map is defined by $g(p) = \phi_1^{-1}(f(p))$, where ϕ_1 maps $U_1 = \mathbf{C}_\infty - \{\infty\}$ homeomorphically to $\mathbf{C}$. For $p \notin P$, we have $g(p) = \infty$. The following proposition relates the multiplicity of g at p (§ 1.3) to the order of f at p.

Proposition: Let $R = (T, A)$ be a Riemann surface, let P be a discrete subset of T, and let $f\colon R - P \to \mathbf{C}$ be a meromorphic function with P as its set of poles. Let $g\colon R \to \mathbf{C}_\infty$ be the associated holomorphic map. Let p be a point of T. Let $\mathit{mult}_p\ g$ denote the multiplicity of g at p.

1. *If $p \in P$, then $\mathit{mult}_p\ g = -\mathit{ord}_p\ f$.*
2. *Otherwise $\mathit{mult}_p\ g = \mathit{ord}_p\ (f - f(p))$.*

Proof: Let p be a point of T. Choose a chart $C = (U, \phi)$ of A centered at p, and let $F = f \circ \phi^{-1}$.

(1) Assume $p \in P$. Then in a punctured neighborhood of zero F has a Laurent series expansion

$$F(z) = \sum_{j=-n}^{\infty} a_j z^j = z^{-n} \sum_{j=0}^{\infty} a_{j-n} z^j = z^{-n} H(z),$$

where $n > 0$, $a_{-n} \neq 0$, $\mathrm{ord}_p\ f = -n$, and H is holomorphic in a neighborhood of zero with $H(0) \neq 0$. Let $C_2 = (U_2, \phi_2)$ be the chart on $\mathbf{C}_\infty$ that maps $\mathbf{C}_\infty - \{0\}$ homeomorphically to $\mathbf{C}$, with $\phi_{12} = \phi_{21} = 1/z$. Let $G = \phi_2 \circ g \circ \phi^{-1}$. Then G maps zero to zero, so the multiplicity at p of g is the order of G at zero. In a punctured neighborhood of zero, we have

$$G = \phi_2 \circ \phi_1^{-1} \circ f \circ \phi^{-1} = \phi_{12} \circ F = 1/F.$$

Therefore in a neighborhood of zero we have $G(z) = z^n(1/H(z)) = z^n P(z)$, where $P(z)$ is a power series expansion with a nonzero constant term. Therefore g has multiplicity n at p, as required.

(2) Assume $p \notin P$. Let $G = \phi_1 \circ g \circ \phi^{-1}$. In a neighborhood of zero we have

$$G = \phi_1 \circ \phi_1^{-1} \circ f \circ \phi^{-1} = f \circ \phi^{-1} = F.$$

The multiplicity of g at p is the order of

$$G - G(0) = f \circ \phi^{-1} - (f \circ \phi^{-1})(0) = f \circ \phi^{-1} - f(p) = (f - f(p)) \circ \phi^{-1}.$$

This is the order at p of $f - f(p)$. □

2. Compact Riemann Surfaces

Recall that a subset S of a topological space T is **compact** if every open cover of S in T has a finite subcover. A topological space T is compact if the entire space is compact as a subset of itself. A Riemann surface $R = (T, A)$ is compact if T is compact.

In this section, we present some basic properties of compact Riemann surfaces.

2.1. The Genus

From the theory of complex manifolds, we know the following:

1. A Riemann surface is an orientable manifold of real dimension two. Here “orientable” means that a local choice of an “up” direction or a “clockwise” orientation of angles, if preserved at every increment along a path on the manifold, is preserved along the entire path. For a non-orientable manifold, e.g., a Möbius strip, this

[2] Note we have in fact shown that the family $\omega = \{\omega_i\}$ of meromorphic one forms on charts C_i of A is a meromorphic one form on R. See *Complex Charts on Topological Surfaces*, § 4.3.

property does not hold.[3]

2. Up to homeomorphism, a compact orientable manifold of real dimension two is a sphere with g handles, or equivalently a torus (donut) with g holes, for $g \geq 0$. See [Massey 1991], I, Theorem 7.2.

Therefore every compact Riemann surface is equivalent to a g-holed torus, for $g \geq 0$. The number g is called the **genus** of the Riemann surface.

The Riemann sphere (see *Complex Charts on Topological Surfaces*, § 5) is a compact Riemann surface of genus zero. One can construct a compact Riemann surface of genus one by specifying a lattice (parallelogram grid) in the complex plane and identifying the pairs of opposite edges of each parallelogram in the lattice. This Riemann surface is called the **complex torus**. See, e.g., [Miranda 1995].

2.2. The Euler Number

From the theory of topology, we know the following:

1. Every compact Riemann surface $R = (T, A)$ has a **triangulation**, i.e., a decomposition of T into closed subsets $\{S_i\}$, in which
   a. Each S_i is homeomorphic to a triangle; and
   b. For each $i \neq j$, S_i and S_j are disjoint, or they meet at a single vertex, or they meet along a single edge.
2. The **Euler number** of a triangulated compact Riemann surface is defined as $v - e + t$, where v is the number of vertices, e is the number of edges, and t is the number of triangles in the triangulation.
3. The Euler number of a compact Riemann surface R is a property of R, independent of the triangulation. See [Miranda 1995] for a sketch of the proof.

With these facts in hand, we can establish the following fundamental result:

Theorem: Let R be a compact Riemann surface of genus g. Then the Euler number of R is $2 - 2g$.

Proof: By induction it suffices to show that (1) a sphere (which has genus $g = 0$) has Euler number 2; and (2) the Euler number decreases by two whenever we add a handle.

(1) We can triangulate a sphere with a tetrahedron, which has 4 vertices, 6 edges, and 4 triangles. So the Euler number is $4 - 6 + 4 = 2$.

(2) We can add a handle by (a) deleting the faces of two triangles and (b) adding cylinder with triangular bases that connects the two triangles. Step (a) removes two triangles. In step (b), by dividing each rectangular face of the cylinder into two triangles, we can triangulate the faces of the cylinder with 6 new triangles and 6 new edges. So the Euler number changes by $-2 - 6 + 6 = -2$ every time we add a handle. □

2.3. Topological Properties

In this section, we establish some basic topological properties of compact Riemann surfaces.

The finite disjointness property: Let T be a topological space. We say that T has the **finite disjointness property** if for every subset S of T and every open set V that contains at most finitely many points of S, every point v in $V - S$ has an open neighborhood W_v that is disjoint from S. The topological space $\mathbf{C}$ has the finite disjointness property, because either (a) $V \cap S = \varnothing$ or (b) we can let W_v be an open ball contained in V whose radius is smaller than the smallest distance from v to a point of $V \cap S$, which is finite. We now show that the topological space of a Riemann surface has the finite disjointness property.

Proposition 1: Let $R = (T, A)$ *be a Riemann surface. The topological space T has the finite disjointness property.*

Proof: Let S be a subset of T, let V be an open set that contains at most finitely many points of S, and let v be a point of $V - S$. Choose a chart $C = (U, \phi)$ containing v, and let $W = U \cap V$. If W contains no points of S, then let $W_v = W$. Otherwise $\phi(W)$ is open, $\phi(S \cap W)$ is finite, $\phi(v) \in \phi(W)$, and $\phi(v) \notin \phi(S \cap W)$. Since $\mathbf{C}$ has the finite disjointness property, we can choose an open neighborhood X of $\phi(v)$ that is disjoint from $\phi(S \cap W)$. Then $W_v = \phi^{-1}(X \cap \phi(W))$ is the required open neighborhood of v. □

[3] The orientability of a Riemann surface follows from fact that holomorphic maps preserve both the magnitude and the orientation of angles. See [Lang 1999], I, § 7. In particular, the transition function between any pair of charts preserves orientation.

Sequential compactness: Let T be a topological space, let I be an infinite subset of the natural numbers, and let $S = \{s_i\}_{i \in I}$ be a sequence of points in T (i.e., a set of points s_i ordered by the ordering of i in the natural numbers).

- We say that S **converges** to a point p in T if for any neighborhood U of p, there exists i in I such that for all $j \geq i$, $s_j \in U$. It is clear from the definition that if T is Hausdorff, then S converges to at most one point (i.e., if S converges to p and to q, then $p = q$).
- We say that a sequence S_2 is a **subsequence** of a sequence S_1 if S_2 may be obtained from S_1 by deleting elements and preserving the order of the remaining elements. For example, $1, 1, 1, \ldots$ is a subsequence of $1, 2, 1, 2, \ldots$.
- We say that a S has a **convergent subsequence** if some subsequence of S converges to a point p in T. For example, the sequence $1, 2, 1, 2, \ldots$ has a convergent subsequence $1, 1, 1, \ldots$. It also has a convergent sequence $2, 2, 2, \ldots$. It also has many other convergent subsequences, consisting of alternating ones and twos followed by all ones or all twos.

Let T be a compact topological space. We say that T is **sequentially compact** if every sequence S in T has a convergent subsequence. We now show that the topological space of a compact Riemann surface is sequentially compact.

Proposition 2: Let $R = (T, A)$ be a compact Riemann surface. The topological space T is sequentially compact.

To prove this proposition, we need two lemmas.

Lemma 1: Let T be a compact topological space and $S \subseteq T$ be a set. If S is closed and discrete, then S is finite.

Proof: Because S is closed, $T - S$ is open. Because S is discrete, we can choose a family of open sets $\{U_s\}_{s \in S}$ such that (1) for each s in S we have $s \in U_s$; and (2) for each s and t in S with $s \neq t$ we have $U_s \cap U_t = \emptyset$. Then $C = (T - S) \bigcup_{s \in S} U_s$ is an open cover of T; since T is compact, it has a finite subcover. But if S is infinite, there is no finite subcover: if we delete any set U_s from C, then there is no way to cover s. Therefore S must be finite. □

Note that Lemma 1 does not hold in general if S is not closed. For example, consider the set $S = \{2^{-n} \mid n \geq 0\} \subseteq \mathbf{R}$. S is a subset of the compact set $[0, 1]$, but it is not closed. It is both infinite and discrete.

Lemma 2: Let T be a compact topological space with the finite disjointness property. Then T is sequentially compact.

Proof: Let S be a sequence of elements in T, and let S also denote the set of points s_i in S. Either S has a convergent subsequence containing infinitely many distinct points, or it does not. If it does, the statement of the lemma is satisfied. Otherwise, we have the following:

1. Each point p in $T - S$ has some neighborhood containing at most finitely many points of S, so by the finite disjointness property, there is an open neighborhood of p contained in $T - S$. Therefore $T - S$ is open, i.e., S is closed.
2. Each point s_i in S has some neighborhood containing at most finitely many points of S, so by the finite disjointness property, S is discrete.

Therefore by Lemma 1, S is finite as a set, and so as a sequence it must have at least one element repeated infinitely. The subsequence consisting of just that element is then a convergent subsequence. □

Proof of the proposition: The result follows from Proposition 1 and Lemma 2. □

3. Holomorphic Maps on Compact Riemann Surfaces

Holomorphic maps on compact Riemann surfaces have several special properties, which we develop in this section.

3.1. Surjectivity and Compactness of the Image

Our first result says that a nonconstant holomorphic map from a compact Riemann surface is surjective, and that the image is compact.

Theorem: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces, and let $f \colon R_1 \to R_2$ be a nonconstant holomorphic map. Suppose that $T_1 \neq \emptyset$ and R_1 is compact. Then (1) R_2 is compact and (2) f is surjective.

To prove this theorem we will need two lemmas. First we need the standard result that a continuous image of a compact set is compact:

Lemma 1: Let T_1 and T_2 be topological spaces, and let $f: T_1 \to T_2$ be a continuous map. Let $S \subseteq T_1$ be a compact set. Then $f(S)$ is compact.

Proof: Let $C = \{U_i\}_{i \in I}$ be an open cover of $f(S)$. Then $D = \{f^{-1}(U_i)\}_{i \in I}$ is an open cover of S. Because S is compact, we can find a finite subcover $\{f^{-1}(U_j)\}_{j \in J}$ of D. Then $\{U_j\}_{j \in J}$ is a finite subcover of C. □

We also need the standard result that a compact subset of a Hausdorff space is closed:

Lemma 2: Let T be a Hausdorff topological space, and let $S \subseteq T$ be a compact set. Then S is closed.

Proof: If $S = T$, then S is closed because T is closed. Otherwise by the definition of a closed set, we need to show that $T - S$ is open. Fix a point $p \in T - S$. It suffices to find an open set X_p such that $p \in X$ and $X_p \subseteq T - S$. Since T is Hausdorff, for each $s \in S$ we can find open sets $U_{p,s}$ and $V_{p,s}$ such that $p \in U_{p,s}$, $s \in V_{p,s}$, and $U_{p,s} \cap V_{p,s} = \emptyset$. Then $\{V_{p,s}\}_{s \in S}$ is an open cover of S, so it has a finite subcover $\{V_{p,i}\}_{i \in I}$ for some finite set $I \subseteq S$. Let $X_p = \bigcap_{i \in I} U_{p,i}$. Then X_p contains p, X_p is open because it is a finite intersection of open sets, and X_p is disjoint from S because it is disjoint from $\bigcup_{i \in I} V_{p,i}$. Therefore X_p is the required open neighborhood of p. □

Proof of the theorem: (1) By § 1.1, Lemma 1, $f: T_1 \to T_2$ is continuous; and by Lemma 1 above, $f(T_1)$ is compact. (2) By § 1.4, $f(T_1)$ is open. By Lemma 2, $f(T_1)$ is closed, so $T_2 - f(T_1)$ is open. By assumption $T_1 \neq \emptyset$, so $f(T_1) \neq \emptyset$. T_2 is connected and $T_2 = f(T_1) \cup (T_2 - f(T_1))$, so $T_2 - f(T_1) = \emptyset$, i.e., $f(T_1) = T_2$. □

3.2. Bounded Holomorphic Functions

Using the results from the previous section, we can show that if a function f is holomorphic and bounded on a compact Riemann surface, then f is constant.

Theorem 1: Let $R = (T, A)$ be a compact Riemann surface, and let $f: R \to \mathbf{C}$ be a holomorphic function. If f is bounded (i.e., there exists a real number $r \geq 0$ such that for all $p \in T$ we have $|f(p)| \leq r$), then f is constant.

Proof: Let $g: R \to \mathbf{C}_\infty$ be the associated holomorphic map to the Riemann sphere (§ 1.5). Then g is a holomorphic map between compact Riemann surfaces. If g were nonconstant, then by § 3.1 it would be surjective. But g is not surjective, because $\infty \notin g(T)$. Therefore g is constant, and so f is constant. □

The analogous statement for holomorphic functions on $\mathbf{C}$ is the classic result in complex analysis called **Liouville's theorem**. The theory of holomorphic maps gives us a simple proof.

Theorem 2 (Liouville's Theorem): Let $f: \mathbf{C} \to \mathbf{C}$ be a holomorphic function. If f is bounded, then f is constant.

Proof: We may interpret f as a holomorphic function on $\mathbf{C}_\infty - \{\infty\}$, with an isolated singularity at ∞. On any chart $C_i = (U_i, \phi_i)$ containing ∞, the local function $f_i = f \circ \phi_i^{-1}$ is bounded, so f_i has a removable singularity at ∞. See *Calculus Over the Complex Numbers*, § 5.2. Therefore f has a removable singularity at ∞, i.e., there is a holomorphic function $g: \mathbf{C}_\infty \to \mathbf{C}_\infty$ such that $g = f$ on $\mathbf{C}$. By Theorem 1, g is constant. Therefore f is constant. □

3.3. Finiteness of the Inverse Image of a Point

Our next result says that, for a nonconstant holomorphic map between compact Riemann surfaces, the inverse image of a point is finite.

Theorem: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be compact Riemann surfaces, and let $f: R_1 \to R_2$ be a nonconstant holomorphic map. Fix a point $q \in T_2$. Then $f^{-1}(q)$ is a finite subset of T_1.

We will need two lemmas.

Lemma 1: Let $R = (T, A)$ be a Riemann surface, and let p be a point of T. Then the set $S = \{p\}$ is closed.

Proof: Choose a chart $C = (U, \phi)$ containing p. Then $\phi(S)$ contains a single point, so it is a closed set in $\mathbf{C}$. $\phi^{-1}: \phi(U) \to U$ is a homeomorphism, which takes closed sets to closed sets, so $S = \phi^{-1}(\phi(S))$ is closed. □

Lemma 2: Let T_1 and T_2 be topological spaces, and let $f: T_1 \to T_2$ be a continuous map. Let $S \subseteq T_2$ be a closed set. Then $f^{-1}(S)$ is closed.

Proof: Because S is closed, $T_2 - S$ is open. Because f is continuous, $f^{-1}(T_2 - S)$ is open. Therefore $f^{-1}(S) = T_1 - f^{-1}(T_2 - S)$ is closed. □

Proof of the theorem: By Lemma 1, $\{q\}$ is closed in T_2. By Lemma 2, $f^{-1}(q)$ is closed in T_1. By § 1.1, $f^{-1}(q)$ is discrete. By § 2.3, Lemma 1, $f^{-1}(q)$ is finite. □

3.4. Finiteness of the Sets of Ramification and Branch Points

We now prove that, for a nonconstant holomorphic map between compact Riemann surfaces, the set of ramification points is finite, as is the set of branch points.

Theorem: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be compact Riemann surfaces, and let $f: R_1 \to R_2$ be a nonconstant holomorphic map. Then (1) the set of ramification points of f is finite; and (2) the set of branch points of f is finite.

Proof: (1) Let S be the set of ramification points of f. Let p be a point of T_1 which is not in S. f behaves like z in an open neighborhood of p, so $T_1 - S$ is open. Therefore S is closed. On the other hand, S is discrete, because for any branch point p, f locally behaves like z^m, so there is a punctured neighborhood of p where f has multiplicity one. Because S is closed and discrete and T_1 is compact, S is finite by § 2.3, Lemma 1.

(2) Since each branch point is the image under f of a ramification point, (2) follows immediately from (1). □

3.5. The Degree of a Holomorphic Map

Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be compact Riemann surfaces, and let $f: R_1 \to R_2$ be a holomorphic map. For each point q in T_1, let $m_f(q)$ denote the multiplicity of f at q (§ 1.3). Define a function $d_f: T_2 \to \mathbf{N}$ as follows:

$$d_f(q) = \sum_{p \in f^{-1}(q)} m_f(p).$$

In English, $d_f(p)$ is the sum of the multiplicities of the points p such that $f(p) = q$. By § 3.3, $f^{-1}(q)$ is finite, so this function is well-defined. We call $d_f(q)$ the **degree** of the map f at the point q.

We claim that d_f is a constant function on T_2. To prove this claim, we need two lemmas.

Lemma 1: Let T_1 and T_2 be topological spaces, and let $f: T_1 \to T_2$ be a continuous map. Let $X = \{x_i\}_{i \in I}$ be a sequence of points in T_1 that converges to a point x. Then the sequence $Y = \{y_i = f(x_i)\}_{i \in I}$ in T_2 converges to $f(x)$.

Proof: Let U be an open neighborhood of $f(x)$. By continuity, $f^{-1}(U)$ is open in T_1. By the convergence of X, for some i in I, x_j lies in $f^{-1}(U)$ for all $j \geq i$. Therefore $f(x_j)$ lies in U for all $j \geq i$. □

Lemma 2: Let T be a topological space, let p be a point of T, and let $S = \{s_i\}_{i \in I}$ be a sequence that converges to a point p. If a subsequence $S_J = \{s_j\}_{j \in J}$ converges to q, then $q = p$.

Proof: Let U be an open neighborhood of p. By the convergence of S, there is an i in I such that for all k in I with $k \geq i$, s_k lies in U as an element of S. But $J \subseteq I$. Thus for all k in J with $k \geq i$, s_k lies in U as an element of S_j. □

Theorem: Let R_1 and R_2 be compact Riemann surfaces, and let $f: R_1 \to R_2$ be a holomorphic map. The function $d_f(p)$ is constant on T_2.

Proof: Choose a point q in T_2, and let $f^{-1}(q) = \{p_i\}_{i \in I}$. For each i let $m_i = m_f(p_i)$; then by § 1.3 there exists a chart $C_{i1} = (U_{i1}, \phi_{i1})$ of A_1 centered at p_i and a chart $C_{i2} = (U_{i2}, \phi_{i2})$ of A_2 centered at q such that

$$F_i = \phi_{i2} \circ f \circ \phi_{i1}^{-1} = z^{m_i}.$$

Let $U_1 = \bigcup_{i \in I} U_{i1}$, and let $U_2 = \bigcap_{i \in I} U_{i2}$. It will suffice to find an open set $V \subseteq U_2$ such that $\phi_2(V)$ is an open ball and $f^{-1}(V) \subseteq U_1$, for in this case we will know the following, for each $v \in V$ and each $i \in I$:

1. If $v = q$, then $f^{-1}(v) \cap U_{i1}$ contains one point with multiplicity m_i.
2. If $v \neq q$, then $f^{-1}(v) \cap U_{i1}$ contains m_i points, each with multiplicity one.

In either case, for each i, the amount contributed to $d_f(u)$ by $f^{-1}(v) \cap U_{i1}$ is the constant value m_i on V. Therefore the amount contributed by $f^{-1}(v)$ is constant on V. Therefore d_f, interpreted as a map to the Hausdorff space $\mathbf{R}$, is both continuous and locally constant, so by § 1.1, Lemma 1, it is constant.

Since I is finite, U_2 is open. Therefore to find the open set V, it suffices to find an open neighborhood W of q such that $f^{-1}(W) \subseteq U_1$; for in this case we can choose V inside the open set $W \cap U_2$. Let y be a point of T_2 for which no such open neighborhood W exists; we will show that $y \neq q$. Fix an i in I and a $J > 0$ such that, for each $j > J$, the open ball $B(\phi_{i2}(y), 1/j)$ lies in $\phi_{i2}(U_2)$. Because W does not exist, we can construct a sequence $Y = \{y_j\}_{j > J}$ such that, for each j, y_j lies in $\phi_{i2}^{-1}(B_j)$ and $f^{-1}(y_j)$ is not contained in U_1. Therefore we can construct a sequence $X = \{x_j\}_{j > J}$ such that $f(x_j) = y_j$ and $x_j \notin U_1$ for each j. By § 2.3, Proposition 2, X has a convergent subsequence

$X_K = \{x_k\}_{k \in K}$. X_K cannot converge to any point in U_1, so it must converge to a point x such that $f(x) \neq q$. By Lemma 1, $Y_K = \{y_k = f(x_k)\}_{k \in K}$ converges to $f(x) \neq q$. But Y converges to y, so by Lemma 2, Y_K converges to y, and therefore $y \neq q$. □

By the theorem, a holomorphic map f between compact Riemann surfaces has the same degree $d_f(p)$ at every point of T_2. We call this number the **degree** of the map f and denote it deg f.

3.6. The Sum of the Orders of a Meromorphic Function

In this section we state and prove a basic result about the sum of the orders of a meromorphic function at the points of a compact Riemann surface.

Let $R = (T, A)$ be a compact Riemann surface, and let f be a meromorphic function on R. Then

$$\sum_{p \in T} \operatorname{ord}_p f = 0.$$

Proof: Let $g\colon R \to \mathbf{C}_\infty$ be the associated holomorphic map, and let p be a point of T. If p is neither a zero nor a pole of f, then there is a chart $C = (U, \phi)$ of A centered at p such that $f \circ \phi^{-1}$ has a power series expansion at zero with a nonzero constant term, so $\operatorname{ord}_p f = 0$. Therefore, writing 0 to denote $\phi_1^{-1}(0) \in \mathbf{C}_\infty$, we have

$$\sum_{p \in T} \operatorname{ord}_p f = \sum_{p \in g^{-1}(0)} \operatorname{ord}_p f + \sum_{p \in g^{-1}(\infty)} \operatorname{ord}_p f.$$

Since $f(p) = 0$ in the first term on the right-hand side, we have

$$\begin{aligned}\sum_{p \in T} \operatorname{ord}_p f &= \sum_{p \in g^{-1}(0)} \operatorname{ord}_p (f - f(p)) + \sum_{p \in g^{-1}(\infty)} \operatorname{ord}_p f. \\ &= \sum_{p \in g^{-1}(0)} \operatorname{mult}_p g - \sum_{p \in g^{-1}(\infty)} \operatorname{mult}_p g \quad \text{(by § 1.5)} \\ &= \deg f - \deg f \quad \text{(by § 3.5)} \\ &= 0.\end{aligned}$$

□

For example, let f be the meromorphic function $1/z$ on the Riemann sphere. Then f has a zero of order one at ∞, a pole of order one at 0, and no other zeros or poles. Therefore the sum of the orders is $1 - 1 = 0$. (Remember that when f has a pole of order n at p, $\operatorname{ord}_p f = -n$).

3.7. Hurwitz's Formula

In this section we present a fundamental result that ties together the topology of compact Riemann surfaces with the theory of holomorphic maps.

Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be compact Riemann surfaces, and let $f\colon R_1 \to R_2$ be a nonconstant holomorphic map. By § 3.1, f is surjective. Let τ_2 be a triangulation of T_2 (§ 2.2). Assume that each branch point of T_2 is a vertex of τ_2, and that each triangle in τ_2 has at most one branch point as a vertex. The set of branch points is finite (§ 3.4), so we can always satisfy these requirements by moving vertices around, and by subdividing triangles. Consider τ_2 as a set of points, and let $\tau_1 = f^{-1}(\tau_2)$.

We claim that τ_1 is a triangulation of T_1. Indeed, let p be a point of T_1, and let m be its multiplicity. Let $C_1 = (U_1, \phi_1)$ be a chart centered on p, and let $C_2 = (U_2, \phi_2)$ be a chart centered on $f(p)$, with $\phi_2 \circ f \circ \phi_1^{-1} = z^m$. Let $\sigma_2 = \tau_2 \cap U_2$, and let $\sigma_1 = f^{-1}(\sigma_2)$. If $m = 1$, then f is one-to-one and continuous on U_1, so σ_1 has the same shape as σ_2, up to moving lines around. If $m > 1$, then by assumption $f(p)$ is a vertex. Assume that e edges meet at $f(p)$ in σ_2. Then by the behavior of z^m, σ_1 is a subset of a triangulation in which me edges meet at p.

The triangulation τ_1 is called the **lifting** of the triangulation τ_2 via the map f. For example, let T_1 and T_2 be the Riemann sphere $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, let $f(z) = z^2$, and let τ_2 be the triangulation of $\mathbf{C}_\infty$ given by three great circles, two that meet perpendicularly at 0 and at ∞, and one that meets the other two perpendicularly at each of the midpoints between 0 and ∞. Then the lifting τ_1 consists of τ_2 plus two more great circles through 0 and ∞, such that the eight edges meeting at each of 0 and ∞ are evenly spaced around those points.

In general, let $d = \deg f$, and let $m(p)$ be the multiplicity of f at each point p in T_1. Observe the following:

1. Let e_i be the number of edges in τ_i. A point q in T_2 with e radiating edges lifts to a set of points $\{p_i\}_{i \in I}$ each with $m(p_i)e$ radiating edges. The total number of radiating edges is de, because $\sum_{i \in I} m(p_i) = d$. Therefore $e_1 = de_2$.
2. Let t_i be the number of triangles in τ_i. For a similar reason, $t_1 = dt_2$.
3. If $m(p) = 1$ at every point p in T_1, then each vertex q of τ_2 lifts to d points in T_1, and $v_1 = dv_2$. For every point p where $m(p) > 1$, p is a vertex of τ_1, and this formula over-counts v_1 by $m(p) - 1$. On the other hand, where $m(p) = 1$, we have $m(p) - 1 = 0$. Therefore in the general case we have
$$v_1 = dv_2 - \sum_{p \in T_1} (m(p) - 1).$$

Now let $E_i = v_i - e_i + t_i$ be the Euler number of T_i. We have
$$\begin{aligned} E_1 &= v_1 - e_1 + t_1 \\ &= (dv_2 - \sum_{p \in T_1} (m(p) - 1)) - de_2 + dt_2 \\ &= d(v_2 - e_2 + t_2) - \sum_{p \in T_1} (m(p) - 1) \\ &= dE_2 - \sum_{p \in T_1} (m(p) - 1). \end{aligned} \tag{1}$$

Finally, let g_i be the genus of T_i. Then $E_i = 2 - 2g_i$, so after negating both sides of (1) we can write
$$\boxed{2g_1 - 2 = d(2g_2 - 2) + \sum_{p \in T_1} (m(p) - 1).} \tag{2}$$

Equation (2) is called **Hurwitz's formula**.

To continue the example of $f(z) = z^2$ on the Riemann sphere, we have $g_1 = g_2 = 0$. The map f has degree 2 with multiplicity 2 at zero and at ∞ and multiplicity one everywhere else. Thus the left-hand side of (2) evaluates to -2, and the right-hand side evaluates to $2 \cdot (-2) + (1 + 1) = -2$.

References

Bocchino, R. *Calculus Over the Complex Numbers. https://rob-bocchino.net/Professional/Diversions.html.*

Bocchino, R. *Complex Charts on Topological Surfaces. https://rob-bocchino.net/Professional/Diversions.html.*

Bocchino, R. *The Inverse and Implicit Mapping Theorems. https://rob-bocchino.net/Professional/Diversions.html.*

Lang, S. *Complex Analysis.* Fourth Edition. Springer Verlag 1999.

Massey, W. *A Basic Course in Algebraic Topology.* Springer Verlag 1991.

Miranda, R. *Algebraic Curves and Riemann Surfaces.* American Mathematical Society 1995.