The Notation $d\bar{z}$ in Complex Analysis

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In discussions of complex analysis, one encounters differential forms such as

$$f\ dz + g\ d\bar{z}.$$

Typically the use of the symbol $d\bar{z}$ is justified by appealing to the formulas

$$z = x + iy \qquad \bar{z} = x - iy$$

$$x = \frac{z+\bar{z}}{2} \qquad y = \frac{z-\bar{z}}{2i}$$

and writing down the following partial derivatives:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}. \tag{1}$$

See, e.g., [Miranda 1995].

While this method has a certain plausibility, it is not obviously justified. The notation seems intentionally ambiguous: does $\bar{z}$ mean, as it does elsewhere, the complex conjugate of the complex number z? If so, how can we treat $\bar{z}$ as an independent variable and write $\partial f/\partial\bar{z}$? If not, how can we write $\bar{z} = x - iy$? Also, the variable $\bar{z}$ does not appear in the definition of the function f. So how is it even valid to take a derivative of f with respect to $\bar{z}$? One gets the sense that important details are being swept under the rug.

The purpose of this paper is to look under the rug and determine what is really going on. This paper builds on my papers *The General Derivative*, *Integration in Real Vector Spaces*, *Calculus over the Complex Numbers*, and *Complex Charts on Topological Surfaces*.

A note on writing derivatives: In this paper, we will use the notation for derivatives developed in *Integration in Real Vector Spaces* and *Calculus over the Complex Numbers*. For example, if $f\colon \mathbf{R}^2 \to \mathbf{C}$ is a differentiable function of the real variables (x, y), then (1) $D_x f$ and $D_y f$ are partial derivatives that take pairs of real numbers to complex numbers, and (2) df is the derivative of f, i.e., the function that takes a pair of real numbers (x, y) to the linear map

$$D_x f(x, y)\ dx + D_y f(x, y)\ dy$$

from $\mathbf{R}^2$ to $\mathbf{C}$. $D_x f$ is an alternate notation for $\partial f/\partial x$.

A note on the chain rule: We will avoid using the "mnemonic" chain rule $df/dy = df/dx \cdot dx/dy$, as represented in the partial derivatives (1). This rule is easier to remember than it is to justify. For example, it is not entirely obvious that the formulas (1) are even valid, because on its face the mnemonic chain rule applies to ordinary derivatives, not partial derivatives. Instead, we will use the general chain rule expressed as a composition of linear maps, as presented in § 7.5 of *The General Derivative*. This rule is easy to justify because it completely general and works exactly the same way in every case. Before reading this paper, you should make sure you understand this version of the chain rule.

1. The Motivation for Using $d\bar{z}$

We begin with the motivation for using $d\bar{z}$ to write differential forms. The basic idea is that using dz and $d\bar{z}$ instead of dx and dy avoids some notational awkwardness.

One forms over $\mathbf{R}^2$: Let f and g be differentiable functions from $\mathbf{R}^2$ to $\mathbf{C}$, and consider the differential one form

$$\omega(x, y) = f(x, y)\ dx + g(x, y)\ dy. \tag{1}$$

ω is a map from $\mathbf{R}^2$ to $L(\mathbf{R}^2, \mathbf{C})$, where as usual $L(\mathbf{R}^2, \mathbf{C})$ is the space of linear maps from $\mathbf{R}^2$ to $\mathbf{C}$. The form (1) straightforwardly extends the forms $\omega\colon \mathbf{R}^2 \to L(\mathbf{R}^2, \mathbf{R})$ that we considered in *Integration in Real Vector Spaces*. We have just replaced $\mathbf{R}$ with $\mathbf{C}$ in the rightmost position. The form (1) also extends the forms

$$\omega(z) = f(z)\ dz \tag{2}$$

that we considered in *Calculus over the Complex Numbers*. Indeed, when integrating the form (2) over a path $\sigma\colon [a, b] \to \mathbf{C}$, we can write

$$\int_\sigma f\ dz = \int_{R \circ R^{-1} \circ \sigma} f\ dz = \int_{R^{-1} \circ \sigma} R^*(f\ dz) = \int_{R^{-1} \circ \sigma} (f \circ R)\ dR = \int_{R^{-1} \circ \sigma} (f \circ R)\ (dx + i\ dy), \tag{3}$$

where $R\colon \mathbf{R}^2 \to \mathbf{C} = (x, y) \mapsto x + iy$ is the rectangular coordinate map. Equation (3) is just integration in rectangular coordinates, as discussed in § 3.4 of *Calculus over the Complex Numbers*. If we write the one form on the right of (3) as

$$\omega = (f \circ R)\ dx + i\ (f \circ R)\ dy,$$

then we see that this one form is a special case of the one form (1). In this document, we will use the symbol $\Omega^1(\mathbf{R}^2)$ to denote the vector space over $\mathbf{C}$ of one forms (1).

Constructing pullbacks: Recall that in the study of complex manifolds, we need to construct pullbacks of the form

$$\phi^*\omega = (f \circ \phi)\phi'\ dz,$$

where $\omega = f\ dz$ is a complex one form, and ϕ is a biholomorphic function defined on some open subset of $\mathbf{C}$. See *Complex Charts on Topological Surfaces*. To employ one forms (1) in this context, we need to develop the analogous construction for these forms. Let us first do this in a straightforward way, using dx and dy. This will be somewhat messy; then we will see how introducing the $d\bar{z}$ notation can help clean up the mess.

Suppose we are given a one form $\omega = f\ dx + g\ dy$ in $\Omega^1(\mathbf{R}^2)$ and a holomorphic function $\phi\colon U \subseteq \mathbf{C} \to \mathbf{C}$. We can't directly pull back ω via ϕ, because the types don't match. Instead, we must pull back via the function

$$\Phi\colon R^{-1}(U) \subseteq \mathbf{R}^2 \to \mathbf{R}^2 = R^{-1} \circ \phi \circ R.$$

Recall that in § 2.4 of *Calculus over the Complex Numbers* we called Φ the **real vector field** associated with the complex function ϕ. We also showed that, because ϕ is holomorphic, the function $\Phi(x, y) = (\Phi_x(x, y), \Phi_y(x, y))$ satisfies the **Cauchy-Riemann equations**, i.e.,

$$D_x\Phi_x = D_y\Phi_y \quad D_x\Phi_y = -D_y\Phi_x.$$

Let us compute the pullback of ω via Φ, i.e., the one form $\Phi^*\omega$. First, note that

$$R^{-1} = z \mapsto \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right). \tag{4}$$

You can see this by plugging in $R(x, y) = x + iy$ for z in the right-hand side of (4); after collecting terms, the result is (x, y). Next, applying (4) in the definition of Φ, we have

$$\Phi = (\Phi_x, \Phi_y) = \left(\frac{\phi \circ R + \overline{\phi \circ R}}{2}, \frac{\phi \circ R - \overline{\phi \circ R}}{2i}\right). \tag{5}$$

Now let us compute the derivatives $d\Phi_x$ and $d\Phi_y$ of the coordinate functions of Φ. First we compute the partial derivatives of the terms in (5). By the chain rule we have

$$D_x(\phi \circ R) = (\phi' \circ R)\ D_xR = (\phi' \circ R)\ D_x(x + iy) = \phi' \circ R$$

$$D_y(\phi \circ R) = (\phi' \circ R)\ D_yR = (\phi' \circ R)\ D_y(x + iy) = i\ (\phi' \circ R). \tag{6}$$

The conjugate function $z \mapsto \bar{z}$ is linear in $\mathbf{R}$, because for any real number r we have

$$\overline{rz} = \bar{r}\bar{z} = r\bar{z}.$$

Therefore, by the rule for the derivative of a composition with a linear map,[1] for any differentiable function $\psi : U \subseteq \mathbf{R} \to \mathbf{C}$ we have

$$d\overline{\psi} = \overline{d\psi}, \tag{7}$$

where we write $\overline{\psi}$ to mean $x \mapsto \overline{\psi(x)}$. Then by (6) and (7) we have

$$\begin{aligned} D_x(\overline{\phi \circ R}) &= \overline{D_x(\phi \circ R)} = \overline{\phi' \circ R} \\ D_y(\overline{\phi \circ R}) &= \overline{D_y(\phi \circ R)} = -i\ (\overline{\phi' \circ R}). \end{aligned} \tag{8}$$

Using (6) and (8) we can compute the derivatives

$$\begin{aligned} d\Phi_x &= D_x\Phi_x\ dx + D_y\Phi_x\ dy \\ &= \frac{\phi' \circ R + \overline{\phi' \circ R}}{2}\ dx + \frac{i(\phi' \circ R) - i\ (\overline{\phi' \circ R})}{2}\ dy \\ &= \frac{\phi' \circ R + \overline{\phi' \circ R}}{2}\ dx + \frac{\phi' \circ R - \overline{\phi' \circ R}}{2}\ i\ dy \\ &= \frac{\phi' \circ R}{2}\ (dx + i\ dy) + \frac{\overline{\phi' \circ R}}{2}\ (dx - i\ dy) \end{aligned} \tag{9}$$

and

$$\begin{aligned} d\Phi_y &= D_x\Phi_y\ dx + D_y\Phi_y\ dy \\ &= \frac{\phi' \circ R - \overline{\phi' \circ R}}{2i}\ dx + \frac{i(\phi' \circ R) + i\ (\overline{\phi' \circ R})}{2i}\ dy \\ &= \frac{\phi' \circ R - \overline{\phi' \circ R}}{2i}\ dx + \frac{\phi' \circ R + \overline{\phi' \circ R}}{2i}\ i\ dy \\ &= \frac{\phi' \circ R}{2i}\ (dx + i\ dy) - \frac{\overline{\phi' \circ R}}{2i}\ (dx - i\ dy). \end{aligned} \tag{10}$$

Now we can compute the pullback $\Phi^*\omega$. From the definition of the pullback, we have

$$\begin{aligned} (\Phi^*\omega)(x, y) &= \omega(\Phi(x, y)) \circ d\Phi(x, y) \\ &= [(f \circ \Phi)\ dx + (g \circ \Phi)\ dy] \circ (d\Phi_x, d\Phi_y). \\ &= (f \circ \Phi)\ d\Phi_x + (g \circ \Phi)\ d\Phi_y. \end{aligned} \tag{11}$$

Putting (9) and (10) together with (11) yields

$$\Phi^*\omega = \left[\frac{f \circ \Phi}{2} + \frac{g \circ \Phi}{2i}\right](\phi' \circ R)\ (dx + i\ dy) + \left[\frac{f \circ \Phi}{2} - \frac{g \circ \Phi}{2i}\right](\overline{\phi' \circ R})\ (dx - i\ dy). \tag{12}$$

Simplifying the notation: Notice the following:

1. We have factored (12) so that the terms $\phi' \circ R$ and $\overline{\phi' \circ R}$ each appear once. This factoring is convenient; it cuts down on the mess that would result if we grouped the factors of dx and dy.

[1] See *The General Derivative*, § 7.6.

2. The factoring in item 1 has caused factors $dx + i\ dy$ and $dx - i\ dy$ to appear.

Notice also that we may factor one forms (1) in the same way. That is, we can write

$$\omega = f\ dx + g\ dy = \left[\frac{f}{2} + \frac{g}{2i}\right](dx + i\ dy) + \left[\frac{f}{2} - \frac{g}{2i}\right](dx - i\ dy). \tag{13}$$

Distributing factors and canceling terms on the right-hand side of (13) yields the left-hand side.

Further, if we write $F = f/2 + g/2i$ and $G = f/2 - g/2i$, then we can write

$$\omega = F\ (dx + i\ dy) + G\ (dx - i\ dy) \tag{14}$$

and

$$\Phi^*\omega = (F \circ \Phi)(\phi' \circ R)\ (dx + i\ dy) + (G \circ \Phi)\overline{(\phi' \circ R)}\ (dx - i\ dy), \tag{15}$$

and these forms are considerably simpler than (12) and (13).

This factoring is the motivation for using $d\bar{z}$: the factor $dx + i\ dy$ becomes dz, and the factor $dx - i\ dy$ becomes $d\bar{z}$. However, in this document we will not introduce the symbol $d\bar{z}$ until we properly explain what it means and why we are justified in using it. We do this in the next section.

2. The Justification for Using $d\bar{z}$

To justify the notation $d\bar{z}$, we need to understand how to convert a differentiable function of the real variables x and y into a differentiable function of the complex variables z and $\bar{z}$. To start, let each of Z_1 and Z_2 be a copy of $\mathbf{C}$, considered as a real vector space. This means that the vectors are complex numbers, addition is in $\mathbf{C}$, the field of scalars is $\mathbf{R}$, and scalar multiplication is multiplication of real numbers r times complex numbers z. Let Z be the real vector space $Z_1 \times Z_2$. Now give each Z_i the same norm as $\mathbf{C}$, i.e., $|z| = \sqrt{z\bar{z}}$, and give Z the same norm as $\mathbf{C}^2$, i.e., $|(z_1, z_2)| = \sqrt{|z_1|^2 + |z_2|^2}$. This construction makes each of Z_1, Z_2, and Z into a normed vector space over $\mathbf{R}$.

Next, consider the subset W of Z given by

$$W = \{(z, \bar{z}) \mid z \in \mathbf{C}\}. \tag{1}$$

That is, W contains all and only pairs of complex numbers (z_1, z_2) such that z_2 is the complex conjugate of z_1, and vice versa. W is also a normed real vector space. It inherits its vector space operations and its norm from Z. W is closed under vector addition because

$$(z_1, \overline{z_1}) + (z_2, \overline{z_2}) = (z_1 + z_2, \overline{z_1} + \overline{z_2}) = (z_1 + z_2, \overline{z_1 + z_2}).$$

W is closed under scalar multiplication, because

$$r(z, \bar{z}) = (rz, r\bar{z}) = (rz, \overline{rz}),$$

since r is a real number.

Observe that W and $\mathbf{R}^2$ are isomorphic as real vector spaces, via the isomorphisms

$$(x, y) \mapsto (x + iy, x - iy) \quad \text{and} \quad (z_1, z_2) \mapsto \left(\frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2i}\right). \tag{2}$$

Observe also that if Y is a normed vector space, $U \subseteq Z$ is an open set, and $f : U \subseteq Z \to Y$ is a differentiable function, then the restriction of f to $V = W \cap U$ is differentiable, and the derivative of the restriction is the restriction of the derivative. That is,

$$d(f|_V) = df|_V. \tag{3}$$

This fact is easily established by applying the restriction to both sides of the definition of the derivative

$$df(z + h) = f(z) + df(z)(h) + o(h),$$

where $z = (z_1, z_2) \in U$, $h = (h_1, h_2) \in Z$, and $z + h \in U$.

We now have all the theory we need to make sense of the notation $d\bar{z}$:

1. Formula (1) explains the use of z and $\bar{z}$ as variables. This use is a kind of shorthand. In fact the variables are elements of the real vector space W, i.e., pairs $w = (z_1, z_2)$ subject to the restriction that $w = (z, \bar{z})$ for some complex number z. The shorthand notation overloads $\bar{z}$ to mean both "the complex conjugate of the complex number z" (its usual meaning in complex analysis) and "the second coordinate of a vector $w \in W$, which is the complex conjugate of the first coordinate" (its specialized meaning in differential forms involving $d\bar{z}$).
2. Formula (2) shows that if we have a function f defined on a subset of $\mathbf{R}^2$, then by composing f with a suitable isomorphism we can convert f to a function defined on a subset of W, and vice versa.
3. Formula (3) explains what is actually meant by taking partial derivatives with respect to variables z and $\bar{z}$. The partial derivatives are taken with respect to the coordinates z_1 and z_2 of the real vector space Z. Then the derivatives are restricted to apply to vectors in W, i.e., vectors (z_1, z_2) in Z such that the complex numbers z_1 and z_2 are mutually conjugate.

Now that we have explained and justified the use of $d\bar{z}$ in the literature, we will not actually use this notation. In my view, whatever benefit this notation may give is outweighed by its potential to confuse. The double meaning of $\bar{z}$ described above is inherently confusing. Also confusing is the use of z to refer to the first coordinate of a vector in W, when z usually denotes a complex number in $\mathbf{C}$. Instead of using z and $\bar{z}$, we will write vectors in W as $w = (z_1, z_2)$. We will reserve z and $\bar{z}$ for their original meanings, i.e., "a complex number in $\mathbf{C}$" and "the complex conjugate of z." If you prefer the $d\bar{z}$ notation, you can mentally replace z_1 with z and z_2 with $\bar{z}$ in the rest of this document.

3. Calculus over the Vector Space W

In this section, we show how to use the real vector space

$$W = \{(z, \bar{z}) \mid z \in \mathbf{C}\}$$

defined in the previous section to express functions, one forms, and operations on them.

3.1. Partial Derivatives

First we use W together with the generalized chain rule to express and compute the partial derivatives (1) in the introduction. Let f be a differentiable function from an open set U in $\mathbf{R}^2$ to $\mathbf{C}$. Let $\psi\colon W \to \mathbf{R}^2$ be the isomorphism

$$\psi(w) = (\psi_1(z_1, z_2), \psi_2(z_1, z_2)) = \left(\frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2i}\right) \tag{1}$$

and let $F = f \circ \psi$. Then F is a differentiable function from a subset of W to $\mathbf{C}$. We wish to compute the partial derivatives $D_1 F$ and $D_2 F$. These are the equivalents, in our notation, of the partial derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$.

By the chain rule, we have

$$\begin{aligned}
D_1 F(w) &= df(\psi(w)) \circ D_1 \psi(w) \\
&= [D_x f(\psi(w))\, dx + D_y f(\psi(w))\, dy] \circ (D_1 \psi_1(w), D_1 \psi_2(w)) \\
&= (D_x f \circ \psi)(D_1 \psi_1) + (D_y f \circ \psi)(D_1 \psi_2) \\
&= \frac{D_x f \circ \psi}{2} + \frac{D_y f \circ \psi}{2i} \\
&= \left[\frac{D_x f}{2} + \frac{D_y f}{2i}\right] \circ \psi.
\end{aligned}$$

A similar computation shows that

$$D_2 F = \left[\frac{D_x f}{2} - \frac{D_y f}{2i}\right] \circ \psi.$$

These results agree with what we said in the introduction, except that here we have composed the partial derivatives with ψ to account for the differing domains of the functions f and F.

3.2. Holomorphic Functions

Next we establish the some basic results about the relationship between holomorphic functions defined on $\mathbf{C}$ and differentiable functions defined on W.

Proposition: Let $U \subseteq \mathbf{C}$ be an open set, and let $\phi: U \to \mathbf{C}$ be a holomorphic function. Let $\pi_1: W \to \mathbf{C}$ be the map

$$(z_1, z_2) \mapsto z_1$$

that projects onto the first coordinate of w, and let $F = \phi \circ \pi_1$. Then on the domain of definition of F,

$$D_1 F = \phi' \circ \pi_1 \quad \text{and} \quad D_2 F = 0.$$

Proof: By the chain rule, we have

$$D_1(\phi \circ \pi_1) = (\phi' \circ \pi_1) D_1 \pi_1 = \phi' \circ \pi_1$$

$$D_2(\phi \circ \pi_1) = (\phi' \circ \pi_1) D_2 \pi_1 = 0.$$

□

Corollary: Let $V \subseteq \mathbf{R}^2$ be an open set, and let $f: V \to \mathbf{C}$ be a differentiable function. Let $G = f \circ \psi$, where $\psi: W \to \mathbf{R}^2$ is the isomorphism (1). If there exists a holomorphic function $\phi: R(V) \to \mathbf{C}$ such that $f = \phi \circ R$, then on the domain of definition of G we have

$$D_1 G = \phi' \circ \pi_1 \quad \text{and} \quad D_2 G = 0.$$

As usual, R refers to the rectangular coordinate map $(x, y) \mapsto x + iy$.

Proof: If ϕ exists, then we have

$$G = \phi \circ R \circ \psi.$$

$R \circ \psi$ is the projection function π_1, so the result follows from the proposition. □

In the literature, you will see this result partially stated as follows: "If f is holomorphic, then $\partial f / \partial \bar{z} = 0$." This statement is correct if we accept the overloading of the symbol f to refer to each of the three functions $f: \mathbf{R}^2 \to \mathbf{C}$, $\phi: \mathbf{C} \to \mathbf{C}$, and $G: W \to \mathbf{C}$.[2]

3.3. One Forms

We now return to the subject discussed in § 1, i.e., complex one forms defined on $\mathbf{R}^2$.

Expressing one forms: From § 2, we know that $\mathbf{R}^2$ and W are isomorphic as real vector spaces. Therefore, $\Omega^1(\mathbf{R}^2)$, the space of one forms $f\ dx + g\ dy$, is isomorphic to $\Omega^1(W)$, the space of one forms $f\ dz_1 + g\ dz_2$. The isomorphisms are given by

$$\omega \mapsto \psi^* \omega: \Omega^1(\mathbf{R}^2) \to \Omega^1(W) \quad \text{and} \quad \omega \mapsto (\psi^{-1})^* \omega: \Omega^1(W) \to \Omega^1(\mathbf{R}^2), \tag{2}$$

where $\psi: W \to \mathbf{R}^2$ is the isomorphism (1). The mappings in (2) are linear. For example, let a be a complex number. Then

$$(\psi^*(a\omega))(w) = (a\omega)(\psi(w)) \circ d\psi(w) = a[\omega(\psi(w)) \circ d\psi(w)] = a(\psi^* \omega)(w).$$

The mappings are also mutual inverses. Indeed, let ω be an element of $\Omega^1(\mathbf{R}^2)$. From the pullback composition lemma in § 2.3 of *Complex Charts on Topological Surfaces*, we have

$$(\psi^{-1})^*(\psi^* \omega) = (\psi \circ \psi^{-1})^* \omega = id^* \omega = \omega.$$

Now let ω be a "well-factored" one-form in $\Omega^1(\mathbf{R}^2)$ of the form shown in equation (14) of § 1, i.e.,

$$\omega = f\ (dx + i\ dy) + g\ (dx - i\ dy). \tag{3}$$

Applying ψ^* to ω yields

[2] This kind of overloading is common in calculus notation. I have come to the conclusion that it is needlessly confusing and should be avoided if possible. See the concluding remarks in § 4.

$$\psi^*\omega = [(f \circ \psi)\ (dx + i\ dy) + (g \circ \psi)\ (dx - i\ dy)] \circ d\psi. \tag{4}$$

We may represent $d\psi$ as a vector of linear maps, as follows:

$$d\psi = \left(\frac{dz_1 + dz_2}{2}, \frac{dz_1 - dz_2}{2i} \right). \tag{5}$$

Composing dx with $d\psi$ yields the first coordinate of (5), and composing dy with $d\psi$ yields the second coordinate. Therefore after carrying out the composition and collecting terms we obtain

$$(dx + i\ dy) \circ d\psi = dz_1 \ \text{ and } \ (dx - i\ dy) \circ d\psi = dz_2. \tag{6}$$

Putting (6) together with (4) yields

$$\psi^*\omega = (f \circ \psi)\ dz_1 + (g \circ \psi)\ dz_2. \tag{7}$$

Thus the one form ω transforms into the one form $\psi^*\omega$ as we expect: the factor $(dx + i\ dy)$ becomes dz_1, and the factor $(dx - i\ dy)$ becomes dz_2.[3]

Integrating over paths: Given a path integral over a one form ω in $\Omega^1(\mathbf{R}^2)$, we can convert it into a path integral over a one form $\psi^*\omega$ in $\Omega^1(W)$ in the usual way:

$$\int_\sigma \omega = \int_{\psi \circ \psi^{-1} \circ \sigma} \omega = \int_{\psi^{-1} \circ \sigma} \psi^*\omega.$$

The path σ maps a real interval $[a, b]$ into $\mathbf{R}^2$, and the path $\psi^{-1} \circ \sigma$ maps $[a, b]$ into W. This technique lets us do path integration over one forms expressed in terms of dz_1 and dz_2, which is notationally convenient.

Constructing pullbacks: Let $\omega = f\ dz_1 + g\ dz_2$ be a one form in $\Omega^1(W)$, and let $\phi\colon U \subseteq \mathbf{C} \to \mathbf{C}$ be a holomorphic function. Let $\pi_1\colon W \to \mathbf{C}$ be the function $(z_1, z_2) \mapsto z_1$ that projects onto the first coordinate of w. Its inverse is $z \mapsto (z, \bar{z})$. Let Φ be the function

$$\Phi\colon \pi_1^{-1}(U) \subseteq W \to W = \pi_1^{-1} \circ \phi \circ \pi_1.$$

We will compute the pullback $\Phi^*\omega$. From the definitions, we have

$$\Phi = (\Phi_1, \Phi_2) = (\phi \circ \pi_1, \overline{\phi \circ \pi_1}).$$

By the proposition of § 3.2, we have

$$D_1(\phi \circ \pi_1) = \phi' \circ \pi_1 \ \text{ and } \ D_2(\phi \circ \pi_1) = 0. \tag{8}$$

We claim that for any differentiable function $F\colon W \to \mathbf{C}$, the following formulas are valid:

$$D_1(\overline{F}) = \overline{D_2 F} \qquad D_2(\overline{F}) = \overline{D_1 F}. \tag{9}$$

To justify (9), take the conjugate of both sides of the definition of the partial derivatives. For example:

$$\begin{aligned} \overline{F(w + (h_1, h_2))} &= \overline{F(w)} + D_1 \overline{F(w)} \cdot \overline{h_1} + \overline{o(h_1)} \\ &= \overline{F(w)} + D_1 \overline{F(w)} \cdot h_2 + \overline{o(h_1)}. \end{aligned}$$

The conjugate operation converts h_1 to h_2, flipping the index of the partial derivative. By (8) and (9) we have

$$D_1(\overline{\phi \circ \pi_1}) = \overline{D_2(\phi \circ \pi_1)} = 0.$$

$$D_2(\overline{\phi \circ \pi_1}) = \overline{D_1(\phi \circ \pi_1)} = \overline{\phi' \circ \pi_1} \tag{10}$$

Using (8) and (10) we can compute the derivatives

$$d\Phi_1 = D_1\Phi_1\ dz_1 + D_2\Phi_1\ dz_2 = (\phi' \circ \pi_1)\ dz_1$$

[3] One may be tempted to argue that because $z_1 = x + iy$ and $z_2 = x - iy$, taking derivatives yields $dz_1 = dx + i\ dy$ and $dz_2 = dx - i\ dy$. This statement, while true, does not establish how or why it is valid to replace $dx + i\ dy$ and $dx - i\ dy$ with dz_1 and dz_2 in a one form. To do that, we have to use the isomorphism $\omega \mapsto \psi^*\omega$, as we did in the text.

$$d\Phi_2 = D_1\Phi_2 \; dz_1 + D_2\Phi_2 \; dz_2 = (\overline{\phi' \circ \pi_1}) \; dz_2. \tag{11}$$

Now we can compute the pullback $\Phi^* \omega$. After the analogous sequence of steps to (11) of § 1, we have

$$(\Phi^* \omega)(w) = (f \circ \Phi) \; d\Phi_1 + (g \circ \Phi) \; d\Phi_2. \tag{12}$$

Putting (11) together with (12) yields

$$(\Phi^* \omega)(w) = (f \circ \Phi)(\phi' \circ \pi_1) \; dz_1 + (g \circ \Phi)(\overline{\phi' \circ \pi_1}) \; dz_2. \tag{13}$$

Notice that (13) corresponds to (15) of § 1 in the way we would expect.

4. Concluding Remarks

My motivation for writing this paper was the nagging suspicion, upon encountering the use of $d\bar{z}$ in the literature, that something deeper was going on and was being overlooked. I wanted to figure out what that was. I think I've done that.

Along the way I've learned several interesting details. For example, before writing this, I guessed that the calculus associated with the notation $d\bar{z}$ must be over a real vector space W of elements $(z, \bar{z})$. But I never knew or suspected that the partial derivative with respect to z_1 of a conjugate is the conjugate of the partial derivative with respect to z_2. Yet this is the essential fact that makes the construction of pullbacks work out in § 3.3.

In the course of writing this paper, I've also learned the following:

1. In mathematical writing, there are two different ways of representing function composition. I'll call them the "algebra way" and the "calculus way," because the algebra way mainly occurs in writing about algebra, and the calculus way mainly occurs in writing about calculus. The algebra way is also prevalent in statically typed functional programming.
2. The algebra way is clear about the domains and ranges of functions. The composition of f and g is $f \circ g$. It is not f.
3. In calculus, one routinely encounters statements to the effect that "The composition of f and g is f," albeit implicitly. This kind of statement is the basis of the mnemonic chain rule

   $$\frac{df}{dy} = \frac{df}{dx} \frac{dx}{dy}.$$

   In order for this rule to make sense, we have to accept that if f is a function on a domain X, then it is also a function on any domain Y that can be mapped into X. Also, symbols like x are overloaded to mean both elements of the domain X and functions $x \colon Y \to X$. We can write $f(x)$, as if x refers to an element of the domain of f; and then we can write $x(y)$, as if x is a function from Y to X.
4. The calculus way of expressing function composition is confusing, especially when the composition crosses vector spaces, as in the case of $\mathbf{R}^2$ and W. It is confusing for a function f of two real variables suddenly to become a function f of two complex variables. This confusion is also unnecessary, because we can write the composition directly and use the generalized chain rule, as I have done in this document. The cost is a few more function symbols, but this seems like an acceptable cost, given the increase in clarity.

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