Complex Charts on Topological Surfaces

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This paper explains the concept of a **complex chart**, i.e., an open subset of a topological space that maps to an open subset of a complex vector space in a well-behaved way. Complex charts let us define topological spaces called **complex manifolds**. We can do calculus over complex manifolds in a way that naturally generalizes calculus over the complex numbers.

In this paper we focus on complex charts on **topological surfaces**, i.e., topological spaces that locally map to the real plane $\mathbf{R}^2$ in a well-behaved way. In this case, a chart maps an open set on a topological surface to an open set in the complex plane, which is topologically equivalent to an open set in $\mathbf{R}^2$. This kind of chart is essential in the study of **Riemann surfaces**, i.e., connected complex manifolds of complex dimension one.[1] By extending the concept of complex charts to higher dimensions, we obtain the theory of general complex manifolds.

This paper assumes that you are familiar with the material covered in my paper *Calculus over the Complex Numbers*. It also assumes basic familiarity with sets and maps. See, e.g., § 2 of my paper *Definitions for Commutative Algebra*.

1. Biholomorphic Functions

We begin with a discussion of biholomorphic functions, which are essential in defining complex charts.

Definition: Let $U \subseteq \mathbf{C}$ be a set, and let $\phi: U \to \phi(U) \subseteq \mathbf{C}$ be a function. We say that ϕ is **biholomorphic** if the following statements are true:

1. ϕ is holomorphic.
2. ϕ is one-to one; equivalently, ϕ has an inverse $\phi^{-1}: \phi(U) \to U$ such that $\phi^{-1} \circ \phi$ is the identity function on U and $\phi \circ \phi^{-1}$ is the identity function on $\phi(U)$.
3. ϕ^{-1} is holomorphic.

More succinctly, ϕ is biholomorphic if it is holomorphic and has a holomorphic inverse.

Examples: Here are some examples of biholomorphic functions ϕ on a set U:

1. The identity function $id = z \mapsto z$. This function maps U to U. It is its own inverse.
2. The translation function $T_a = z \mapsto z - a$, where a is a complex number. This function maps U to the set of all $z - a$ such that z is in U. Its inverse is $T_{-a} = z \mapsto z + a$.
3. The function $\phi = z \mapsto z^2$, if U is a disc that does not intersect the origin $z = 0$. ϕ maps U to the set of all z^2 such that z is in U. To see that ϕ is biholomorphic, note the following:
    a. If there are two distinct nonzero complex numbers a and b such that $a^2 = b^2$, then we must have $a = re^{i\theta}$ and $b = re^{i(\theta+\pi)}$, for some $r > 0$ and some angle θ.
    b. No such pair of numbers can lie in a disc that does not intersect the origin. Therefore $\phi^{-1}(z) = z^{1/2}$ is well-defined on U, and it is holomorphic with derivative $\dfrac{z^{-1/2}}{2}$.

Properties: From the definition of a biholomorphic map, we obtain the following properties for every biholomorphic map ϕ:

[1] You may ask: Why are Riemann surfaces called "surfaces" if they have dimension one? Don't surfaces have dimension two? The answer is that in a complex vector space the topological dimension is the number of *real* coordinates, which is two times the complex dimension. For example, the complex numbers have complex dimension one but topological dimension two.

1. *Both ϕ and ϕ^{-1} are continuous.* This fact holds because a differentiable function is continuous.
2. *Both ϕ and ϕ^{-1} are open mappings, i.e., they take open sets to open sets.* Let $V \subseteq \phi(U)$ be an open set. In § 3.1 we will prove that the inverse image of an open set under a continuous function is open. Accepting this fact for now, we see that $\phi^{-1}(V)$ is open because ϕ is continuous. Let $W \subseteq U$ be an open set. $\phi(W)$ is open because $\phi = (\phi^{-1})^{-1}$, so $\phi(W)$ is the inverse image of W under the continuous map ϕ^{-1}.

Composition: Let $\phi\colon U \to \phi(U) \subseteq \mathbf{C}$ and $\psi\colon V \to \psi(V) \subseteq \mathbf{C}$ be biholomorphic functions, with $\phi(U) \subseteq V$. Then $\psi \circ \phi$ is well-defined and biholomorphic, with inverse $\phi^{-1} \circ \psi^{-1}$. $(\psi \circ \phi)(U) = \psi(\phi(U))$ is a subset of $\psi(V)$. See Figure 1.

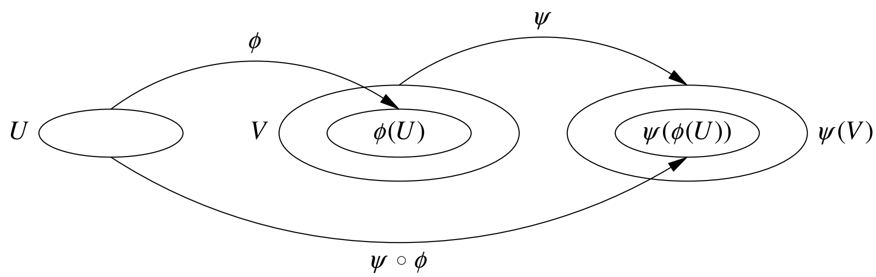

Figure 1: Composition of biholomorphic functions.

By the chain rule, the derivative of $\psi \circ \phi$ is

$$d(\psi \circ \phi)(z) = d\psi(\phi(z)) \circ d\phi(z) = (\psi'(\phi(z))\ dz) \circ (\phi'(z)\ dz) = \psi'(\phi(z))\phi'(z)\ dz.$$

Using the pullback notation for functions and for differential one forms,[2] we can also write

$$d(\phi^* \psi) = \phi^*\ d\psi.$$

2. Bioholomorphic Charts

We now present an important special case of complex charts: biholomorphic charts in the complex plane.

2.1. Basic Definitions

Charts: A **biholomorphic chart** (chart for short, when there is no ambiguity) is a pair $C = (U, \phi)$ consisting of an open set $U \subseteq \mathbf{C}$ and a biholomorphic function $\phi\colon U \to \phi(U) \subseteq \mathbf{C}$. We call U the **chart domain** associated with C. We call ϕ the **chart map** associated with C. We call $\phi(U)$ the **chart image** associated with C.

Let $a \in U$ be a complex number. If $\phi(a) = 0$, then we say that the chart (U, ϕ) is **centered at** a.

We may think of a biholomorphic chart as a change of complex coordinate (or of the corresponding real coordinates). For example, let $U \subseteq \mathbf{C}$ be an open set, let $b \in U$ be a complex number, and let ϕ be the translation map $T_b = z \mapsto z - b$. Then (U, ϕ) is a chart centered at b. It translates a neighborhood of b to a neighborhood of the origin. In *Calculus over the Complex Numbers*, we used this chart to write a power series expansion at b as $P \circ T_b$, where P is a power series expansion at zero.

Atlases: A **biholomorphic atlas** (atlas for short, when there is no ambiguity) is a family of charts $A = \{C_i = (U_i, \phi_i)\}_{i \in I}$, where I is an index set, and the sets U_i cover the complex plane, i.e., $\bigcup_{i \in I} U_i = \mathbf{C}$. For example:

1. Let $I = \{0\}$, and let $C_0 = (\mathbf{C}, id_{\mathbf{C}})$, where $id_{\mathbf{C}}$ is the identity function on $\mathbf{C}$. Then the family of charts consisting of the single element C_0 is an atlas.
2. Let C_0 be as in the previous example. Let C_1 be the chart consisting of the open ball $B(0, 1)$, together with the identity function on $B(0, 1)$. Then $\{C_0, C_1\}$ is an atlas.

[2] Recall that when f and g are functions, we have $g^* f = f \circ g$. When g is a differentiable function and $\omega = f\ dz$ is a one form, we have $(g^*\omega)(z) = \omega(g(z)) \circ dg = f(g(z))g'(z)\ dz$.

Let A_1 and A_2 be complex atlases. We say that A_1 is **included in** A_2 if every chart of A_1 is a chart of A_2. In this case we write $A_1 \subseteq A_2$. For example, the atlas in the first example given above is included in the atlas in the second example given above.

The maximal atlas: The **maximal atlas** is the unique biholomorphic atlas consisting of all biholomorphic charts. It is maximal in the sense that it includes all other biholomorphic atlases on **C**. The maximal biholomorphic atlas exists by Zorn's lemma,[3] and it is unique up to the choice of index set, which is unique up to a bijection.

Note that the maximal atlas contains a very large number of charts. For every open set U in the complex plane, there is a chart for each biholomorphic function ϕ with domain U.

Transition functions: Let A be a biholomorphic atlas on **C**, and let $C_i = (U_i, \phi_i)$ and $C_j = (U_j, \phi_j)$ be charts of A. We define the **transition function** $\phi_{ij} = \phi_j \circ \phi_i^{-1}$. Here ϕ_i^{-1} means the inverse of ϕ_i restricted to the domain $\phi_i(U_i \cap U_j)$ where the composition makes sense.[4] The transition function ϕ_{ij} is defined on $\phi_i(U_i \cap U_j)$, and its image is $\phi_j(U_i \cap U_j)$. It translates the chart image of C_i to the chart image of C_j on the area of overlap between the charts. See Figure 2.

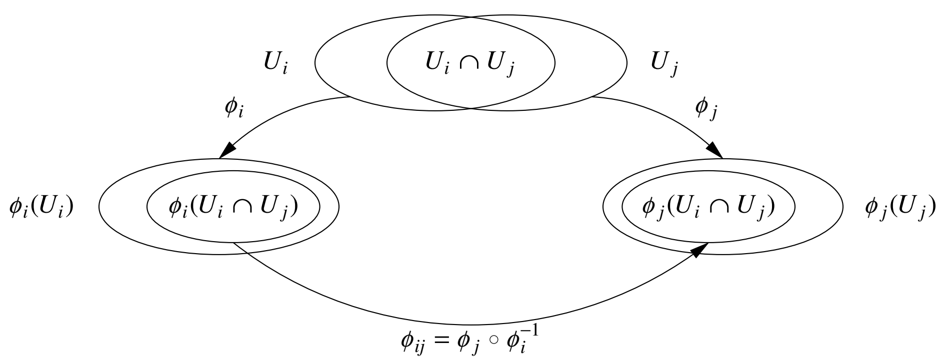

Figure 2: The transition function ϕ_{ij}.

Note the following:

1. ϕ_{ii} is the identity map on $\phi_i(U_i)$.
2. $\phi_{ij}^{-1} = \phi_{ji}$.
3. If $U_i \cap U_j = \varnothing$, then ϕ_{ij} is the trivial function that maps no elements.
4. By the observations in § 1, ϕ_{ij} is biholomorphic.

Charts on open sets: Let $V \subseteq$ **C** be an open set. We use the same definitions given above, after replacing **C** with V, to put charts on V:

1. A biholomorphic chart on V is a pair $C = (U, \phi)$ consisting of an open set $U \subseteq V$ and a biholomorphic function $\phi\colon U \to \phi(U)$.
2. A biholomorphic atlas on V is a family $A = \{C_i = (U_i, \phi_i)\}_{i \in I}$ of charts on V such that the sets U_i cover V.
3. The maximal biholomorphic atlas on V is the unique atlas containing all charts on V.
4. A transition function ϕ_{ij} between charts C_i and C_j on V is as defined above.

Note that **C** itself is an open subset of **C**, so these definitions extend the previous ones.

2.2. Functions

In this section we investigate the behavior of complex functions expressed in terms of biholomorphic charts.

Meromorphic functions: For any set $W \subseteq$ **C** and meromorphic function g on W, we can derive a holomorphic function f on V, where $V = W - P$, P is a discrete set of poles of g on W, and $f = g$ on V. Thus, for the remainder of this section, we will focus on holomorphic functions. The theory presented here applies equally to meromorphic

[3] For an introduction to basic set theory concepts, including Zorn's lemma, see § 2 of my paper *Definitions for Commutative Algebra*.

[4] If we wrote out the domain restriction explicitly, the definition would look like this: $\phi_{ij} = \phi_j \circ (\phi_i^{-1}|_{\phi_i(U_i \cap U_j)})$. This definition is very precise, but also very unwieldy. So we will leave the restriction implicit.

functions, if we always convert a meromorphic function g on W to a holomorphic function f on V in this manner. The advantage of doing this is that f is defined everywhere on its domain, i.e., it is a function $f: V \to \mathbf{C}$.

Charts on the domain: Fix an open set $V \subseteq \mathbf{C}$, an atlas $A = \{C_i\}_{i \in I}$ on V, and a complex function $f: V \to \mathbf{C}$. We express f in terms of the charts C_i of A as follows. For each chart $C_i = (U_i, \phi_i)$, we define $f_i: \phi_i(U_i) \to \mathbf{C}$ to be

$$f_i = (\phi_i^{-1})^* f = f \circ \phi_i^{-1}.$$

We call f_i the **local function** of f with respect to the chart C_i. Because the chart functions ϕ_i are holomorphic, each local function f_i is holomorphic if and only if f is holomorphic.

The following is a basic property of local functions:

For each pair of charts (C_i, C_j), on $\phi_i(U_i \cap U_j)$ we have

$$f_i = \phi_{ij}^* f_j. \tag{1}$$

Proof: On $\phi_i(U_i \cap U_j)$ we have

$$\phi_{ij}^* f_j = \phi_{ij}^*((\phi_j^{-1})^* f) = (\phi_j^{-1} \circ \phi_{ij})^* f = (\phi_i^{-1})^* f = f_i.$$

□

Thus we may think of a complex function $f: V \to \mathbf{C}$ as a family of local functions $f_i: \phi_i(U_i) \to \mathbf{C}$, one for each chart C_i, such that property (1) holds.

Conversely, let $\{f_i\}$ be a family of local functions that satisfy property (1). We may define a function f on V as follows:

$$f(z) = (\phi_i^* f_i)(z),$$

where z is a point in V, and C_i is any chart containing z. By the definition of the atlas A, at least one such chart exists. Further, if C_j is any other chart of A containing z, then at z we have

$$\phi_i^* f_i = \phi_i^*(\phi_{ij}^* f_j) = (\phi_{ij} \circ \phi_i)^* f_j = \phi_j^* f_j.$$

Therefore $f(z)$ is independent of the chart, and so f is well-defined on V. Thus we see that there is a one-to-one correspondence between functions f on V and families of local functions $\{f_i\}$ on the charts of A.

Charts on the image: Fix open sets $V \subseteq \mathbf{C}$ and $W \subseteq \mathbf{C}$ and a complex function $f: V \to W$. Let $A = \{C_i\}_{i \in I}$ be an atlas on V, and let $B = \{C_j\}_{j \in J}$ be an atlas on on W. We express f in terms of the charts of A and B as follows. For each pair (C_i, C_j), where C_i is a chart of A and C_j is a chart of B, we define

$$f_{ij} = \phi_j \circ f_i = \phi_j \circ f \circ \phi_i^{-1}.$$

f_{ij} is defined on $\phi_i(U_i \cap f^{-1}(U_j))$. Its image is $\phi_j(f(U_i \cap f^{-1}(U_j)))$. f is holomorphic if and only if each f_{ij} is holomorphic.

2.3. Differential Forms

In this section, we show how to express complex differential forms in terms of biholomorphic charts. For an introduction to differential forms, see my papers *Integration in Real Vector Spaces* and *Calculus over the Complex Numbers*. Let V be an open subset of $\mathbf{C}$, let A be an atlas of V, and let $C_i = (U_i, \phi_i)$ and $C_j = (U_j, \phi_j)$ be charts of A. For the reasons discussed in § 2.2, we focus on holomorphic functions and one forms on V and on the charts of A.

Local zero forms: First we describe function evaluation as integration of a zero form (i.e., a function) over a zero-dimensional region (i.e., a point). We start with the concept of a local zero form, i.e., a function defined on a chart.

We define the following with respect to the chart C_i of the atlas A:

1. A **local zero form** on C_i is a local function, i.e., a function $f_i: \phi_i(U_i) \to \mathbf{C}$.
2. A **local zero-dimensional region** or **local point** on C_i is a function $\sigma_i: \{0\} \to \phi_i(U_i)$ that maps the single real number 0 to a complex number $\sigma_i(0) = a$ in the chart image $\phi_i(U_i)$.

We define the integral of a local zero form f_i at a local point σ_i as follows:

$$\int_{\sigma_i} f_i = \int_0 \sigma_i^* f_i = \int_0 f_i \circ \sigma_i = f_i(\sigma_i(0)) = f_i(a).$$

Zero forms: Now we define a zero form, i.e., a function defined on an atlas A.

1. A **zero form** on A is a family $f = \{f_i\}_{i \in I}$ of local functions, one for each chart of A, such that property (1) holds. By the observations in § 2.2, f is a function on V.
2. A **zero-dimensional region** or **point** on A is a function $\sigma\colon \{0\} \to V$ that maps the single real number 0 to a complex number a in V.

We define the integral of a zero form f at a point σ as follows:

$$\int_{\sigma} f = \int_{\sigma_i} f_i,$$

where $C_i = (U_i, \phi_i)$ is any chart of A such that U_i contains $\sigma(0)$, and $\sigma_i = \phi_i \circ \sigma$. By the definition of the atlas A, at least one such chart C_i exists. Further, if C_j is any other chart of A such that U_j contains $\sigma(0)$, then

$$\int_{\sigma_j} f_j = \int_{\phi_{ij} \circ \phi_{ji} \circ \sigma_j} f_j = \int_{\phi_{ji} \circ \sigma_j} \phi_{ij}^* f_j = \int_{\sigma_i} \phi_{ij}^* f_j = \int_{\sigma_i} f_i,$$

where the last equality follows from property (1). Therefore the value of the integral does not depend on the choice of chart.

Local one forms: We now discuss the concept of a local one form, i.e., a one form on a chart. We define the following with respect to the chart C_i of the atlas A:

1. A **local one form** ω_i on C_i is a mapping from $\phi_i(U_i)$ to $L(\mathbf{C}, \mathbf{C})$, the space of linear maps from $\mathbf{C}$ to $\mathbf{C}$. We write $\omega_i(z) = f_i(z)\ dz$, where f_i is a function from $\phi_i(U_i)$ to $\mathbf{C}$, and dz is the identity map $z \mapsto z$. For any complex number a in $\phi_i(U_i)$, $\omega_i(a) = f_i(a)\ dz$ is the linear map that takes each complex number b to the complex number $f_i(a)b$.
2. A **local one-dimensional region** or **local path** on C_i is a differentiable mapping $\sigma_i\colon s_i \to \phi_i(U_i)$, where $s_i = [a_i, b_i]$ is an interval of the real line.

We define the integration of a local one form $\omega_i = f_i\ dz$ over a local path σ_i as follows:

$$\int_{\sigma_i} \omega_i = \int_a^b \sigma_i^* \omega_i = \int_a^b \omega_i(\sigma_i(t))\sigma_i'(t)\ dt.$$

Note that if we forget the chart structure of A and think of $\phi_i(U_i)$ as an open set $W \subseteq \mathbf{C}$, then these definitions agree with the definitions that we gave in *Calculus over the Complex Numbers* for a one form on W and for the integration of a one form over a path σ in W.

One forms: Now we define a one form on an atlas A.

1. A **one form** on A is a family $\omega = \{\omega_i\}_{i \in I}$ of local one forms, one for each chart of A, such that for each pair of charts (C_i, C_j), on $\phi_i(U_i \cap U_j)$ we have

$$\boxed{\omega_i = \phi_{ij}^* \omega_j. \qquad (2)}$$

2. A **one-dimensional region** or **path** on A is a mapping $\sigma\colon s \to V$, where $s = [a, b]$ is an interval of the real line, and for every chart $C_i = (U_i, \phi_i)$, such that the image of σ is contained in U_i, the mapping $\sigma_i = \phi_i \circ \sigma\colon [a, b] \to \phi_i(U_i)$ is differentiable.

Fix a one form ω, a path σ, and a chart C_i such that the image of σ is contained in U_i. We define the integral of ω over σ as follows:

$$\int_{\sigma} \omega = \int_{\sigma_i} \omega_i.$$

If C_j is any other chart such that U_j contains the image of σ_j, then we have

$$\int_{\sigma_j} \omega_j = \int_{\phi_{ij} \circ \phi_{ji} \circ \sigma} \omega_j = \int_{\phi_{ji} \circ \sigma_j} \phi_{ij}^* \omega_j = \int_{\sigma_i} \phi_{ij}^* \omega_j = \int_{\sigma_i} \omega_i,$$

where the last equality follows from property (2). Therefore the value of the integral does not depend on the choice of chart.

To integrate over a path that spans multiple chart domains, we divide the path into a **chain** γ of paths, each of which is contained in a chart domain, and we sum the integrals of the paths in the chain. For example, see the chain $\gamma = \sigma_1 + \sigma_2$ shown in Figure 3. The chain spans the chart domains U_1 and U_2, but each σ_i is contained in the chart domain U_i. By definition, $\int_\gamma \omega = \int_{\sigma_1} \omega + \int_{\sigma_2} \omega$.

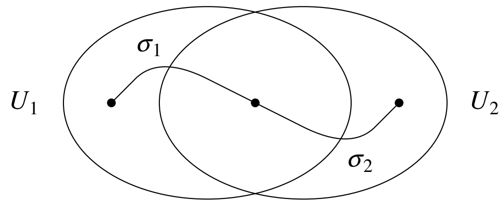

Figure 3: A chain $\gamma = \sigma_1 + \sigma_2$ that spans chart domains U_1 and U_2.

Global one forms: We will call a one form on V a **global one form**, to distinguish it from the local one forms on the charts of A. Notice that we now have two notions of a global one form:

1. A one form on V, i.e., a one form $\omega = f\ dz$ as defined in *Calculus over the Complex Numbers*.
2. A one form on A, i.e., a family of local one forms $\{\omega_i\}$ defined on the charts of A and satisfying property (2).

These concepts are related in a way that is analogous to the relationship between a function f on V and a family of local functions $\{f_i\}$ on A (§ 2.2).

We will need the following results:

Lemma (pullback distribution): Let $\omega = f\ dz$ be a one form defined on a subset of $\mathbf{C}$, and let g and h be holomorphic functions, each defined on a subset of $\mathbf{C}$. Then on the domain where $g^*(h\omega)$ is defined, we have

$$g^*(h\omega) = (g^*h)(g^*\omega).$$

Proof: For all z where $(g^*(h\omega))(z)$ is defined, we have

$$\begin{aligned}(g^*(h\omega))(z) &= (h(g(z))\omega(g(z))) \circ dg(z)\\ &= h(g(z))(\omega(g(z)) \circ dg(z))\\ &= (g^*h)(z)(g^*\omega)(z)\\ &= ((g^*h)(g^*\omega))(z).\end{aligned}$$

□

Lemma (pullback composition): Let $\omega = f\ dz$ be a one form defined on a subset of $\mathbf{C}$, and let g and h be holomorphic functions, each defined on a subset of $\mathbf{C}$. Then on the domain where $g^*(h^*\omega)$ is defined, we have

$$g^*(h^*\omega) = (h \circ g)^*\omega.$$

Proof: For all z where $(g^*(h^*\omega))(z)$ is defined, we have

$$\begin{aligned}(g^*(h^*\omega))(z) &= g^*(\omega(h(z)) \circ dh(z))\\ &= \omega(h(g(z))) \circ dh(g(z)) \circ dg(z)\\ &= \omega((h \circ g)(z)) \circ d(h \circ g)(z)\\ &= ((h \circ g)^*\omega)(z).\end{aligned}$$

□

Now let $\omega = f\ dz$ be a one form on V. Define

$$\omega_i = (\phi_i^{-1})^* \omega.$$

Letting z be the identity function, and writing $z_i = (\phi_i^{-1})^* z = \phi_i^{-1}$, we have

$$\omega_i = (\phi_i^{-1})^*(f\ dz) = ((\phi_i^{-1})^* f)((\phi_i^{-1})^* dz) = f_i\ d((\phi_i^{-1})^* z) = f_i\ dz_i.$$

Then for any charts C_i and C_j, and at any point a such that $(\phi_{ij}^* \omega_j)(a)$ is defined, we have

$$\phi_{ij}^* \omega_j = \phi_{ij}^*((\phi_j^{-1})^* \omega) = (\phi_j^{-1} \circ \phi_{ij})^* \omega = (\phi_i^{-1})^* \omega = \omega_i.$$

Therefore the family $\omega = \{\omega_i\} = \{f_i\ dz_i\}$ is a one form on A.

Conversely, let $\{\omega_i\}$ be a family of local one forms satisfying property (2). Define a one form ω on V as follows:

$$\omega(z) = (\phi_i^* \omega_i)(z),$$

where z is a point in V, and C_i is any chart containing z. If C_j is any other chart of A containing z, then

$$\phi_i^* \omega_i = \phi_i^*(\phi_{ij}^* \omega_j) = (\phi_{ij} \circ \phi_i)^* \omega_j = \phi_j^* \omega_j.$$

Therefore $\omega(z)$ is independent of the chart, and ω is well-defined on V. Thus there is a one-to-one correspondence between one forms on V and one forms on A.

3. Topological Spaces

We wish to generalize the idea of an atlas of biholomorphic charts on $\mathbf{C}$ to an atlas of continuous charts on a topological surface. In this section we discuss general topological spaces, including topological surfaces.

3.1. Definitions

The Euclidean topology: Recall that the Euclidean topology on the n-dimensional real space $\mathbf{R}^n$ is defined as follows:

1. The **distance** $|a - b|$ from a point $a = (a_1, \ldots, a_n)$ to a point $b = (b_1, \ldots, b_n)$ is $\sqrt{(a_1 - b_1)^2 + \ldots + (a_n - b_n)^2}$.
2. The **open ball** $B(a, r)$ with center a and radius r, for $r > 0$, is the set of all points x whose distance to a is less than r, i.e., $|x - a| < r$.
3. The **open sets** in $\mathbf{R}^n$ are unions of open balls.

It is easy to show that a set $U \subseteq \mathbf{R}^n$ is open if and only if, for each point $p \in U$, there is a real number $\delta > 0$ such that $B(p, \delta) \subseteq U$.

The Euclidean topology on the complex numbers $\mathbf{C}$ is the Euclidean topology on $\mathbf{R}^2$, where we identify $\mathbf{C}$ with $\mathbf{R}^2$ via the rectangular coordinate map R. See § 1.2.1 of *Calculus over the Complex Numbers*.

Axiomatic generalization: An **axiomatic generalization** of a concept C characterizes C as a specific instance of a more general and abstract concept A defined by certain axioms, which C also satisfies. Axiomatic generalization is ubiquitous in higher mathematics. For example, the concept of a ring is an axiomatic generalization of the integers.

Topological spaces: A **topological space** is an axiomatic generalization of the Euclidean topology on $\mathbf{R}^n$. It is a pair $T = (S, O)$, where S is a set, and O is a set of subsets of S satisfying the following axioms:

1. The empty set and S are elements of O.
2. Any union of elements of O is an element of O.
3. Any intersection of finitely many elements of O is an element of O.

The set O is called a **topology** on S. The elements of O are called the **open sets** of the topology.

The Euclidean topology on $\mathbf{R}^n$ is a topological space $T = (S, O)$, where $S = \mathbf{R}^n$, and O is the set of unions of open balls in $\mathbf{R}^n$.

We often identify a topological space $T = (S, O)$ with its underlying set S. For example:

- We may write S to refer to T. In this case the topology on S is implied.

- We may write T to refer to S. For example, we may say that a set U is a subset of T, meaning that it is a subset of S.

Complements and closed sets: Fix a topological space $T = (S, O)$, and let U be a subset of S.

1. The **complement** of U, written U^C, is the set difference $S - U$.
2. U is **closed** if and only if its complement U^C is open.

Bases and subbases: Fix topological space $T = (S, O)$ and a set of subsets B of S.

1. B is a **base** or **basis** for the topology O if every element of O may be represented as the union of elements of B. The open balls in $\mathbf{R}^n$ are a basis for the Euclidean topology on $\mathbf{R}^n$.
2. B is a **subbase** or **subbasis** for O if there exists a basis B' for O such that every element of B' except S may be represented as the intersection of finitely many elements of B.

If B is a subbasis for O, then we say that B **generates** O. In this case O is minimal among the topologies on S that contain B.

A set is **countable** if it can be put in one-to-one correspondence with the integers. If T has a countable basis, then we say that T is **countable**. If T has a countable subbasis, then we say that T is **second countable**.

Neighborhoods: Let $T = (S, O)$ be a topological space and p be a point of S. A **neighborhood** of p is a set $U \subseteq S$ that contains p. If U is open (i.e., $U \in O$), then we say that U is an **open neighborhood** of p.

Hausdorff topologies: Fix a topological space T. If for every pair of points p and q in T there exists an open neighborhood U_p of p and an open neighborhood U_q of q such that $U_p \cap U_q = \emptyset$, then we say that T is **Hausdorff**.

The subset topology: Let $T = (S, O)$ be a topological space, and let U be a subset of S. We use the following rule to define a topological space $T' = (U, O')$:

A set $V \subseteq U$ is open in T' if and only if there exists an open set W in T such that $V = U \cap W$.

It is an easy exercise to check that this definition satisfies the axioms for a topology on U.

We call the topology defined above the **subset topology** on U with respect to the topology T. We also call it the topology **induced** on U as a subset of S with respect to T. As an example, the Euclidean topology on the real line $\mathbf{R}$ is the topology induced on $\mathbf{R}$ with respect to the Euclidean topology on $\mathbf{R}^2$.

Compactness: Let $T = (S, O)$ be a topological space, and let U be a subset of S. A **cover** of U is a family of sets $C = \{V_i\}_{i \in I}$ that cover U, i.e., such that $\cup_{i \in I} V_i = U$. An **open cover** of U is a cover of U such that every set V_i is open in T. A **subcover** of a cover C of U is a cover C' of U such that every set in the cover C' is a set in the cover C.

A subset U of S is **compact** with respect to T if every open cover of U has a finite subcover. For example:

1. *A closed interval $[a, b]$, for $a < b$ in $\mathbf{R}$, is compact with respect to the Euclidean topology on $\mathbf{R}$.*

   Proof: Let C be an open cover of $[a, b]$, not necessarily finite. Let S be the set of real numbers $a \le x \le b$ such that $[a, x]$ is covered by a finite number of sets of C. S is not empty, because we can take $x = a$. Therefore by a basic property of the real numbers, S has a least upper bound. It suffices to show that b is the least upper bound of S, and for this it suffices to show that any number r, $a \le r < b$, is not the least upper bound. Fix such a number r. We know that $[a, r]$ is covered by a finite set F open sets of C. The union U of those sets is open, so there exists an open ball $B(r, \varepsilon) \subseteq U$, and we may choose ε so that $B(r, \varepsilon) \subseteq [a, b]$. But then F covers $[a, r + \varepsilon/2]$, so r is not an upper bound, as required. □

2. *An open interval (a, b), for $a < b$ in $\mathbf{R}$, is not compact with respect to the Euclidean topology on $\mathbf{R}$.*

   Proof: It suffices to give an infinite open cover C of (a, b) that has no finite subcover. Define C as follows:

   - The index set I is the set of real numbers in the interval (a, b).
   - The set U_i in C is the open ball (i.e., open interval) with center i and radius half the distance from i to the nearer of the points a and b.

   Clearly C is an open cover of (a, b). Let F be any finite subset of C. Then there is a smallest number i such that $U_i \in F$. Let $c = a + (i - a)/2$. Then the numbers in the interval (a, c) are not covered by U_i, nor are they covered by any U_j, $j > i$. Therefore F is not an open cover of (a, b). □

These statements are special cases of the Heine-Borel theorem, which we state in § 3.2.

Connected sets and spaces: Let $T = (S, O)$ be a topological space, and let U be a subset of S. U is **connected** in T if U cannot be represented as the union of two sets V and W, each of which is a subset of S and is open in T, such that $V \cap W = \emptyset$. For example, the open ball $B(0, 1)$ is connected in $\mathbf{R}^2$, because any pair of open sets V and W such that $V \cup W = B(0, 1)$ must have non-empty intersection. The union of open balls $B(-1, 1) \cup B(1, 1)$ is not connected in $\mathbf{R}^2$, because the open balls have empty intersection.

Let $T = (S, O)$ be a topological space. We say that T is **connected** if S is a connected set in T.

Maps: Let $T_1 = (S_1, O_1)$ and $T_2 = (S_2, O_2)$ be topological spaces. A **map** $f : T_1 \to T_2$ from T_1 to T_2 is a mapping of sets $f : S_1 \to S_2$.

Let $f : T_1 \to T_2$ be a map.

1. We say that f is **continuous** if for every open set U in $f(S_1)$, the set $f^{-1}(U)$ is open in S_1.
2. We say that f is **open** if for every open set U in T_1, $f(U)$ is open in T_2.
3. An **inverse map** for f is a map $f^{-1} : f(S_1) \to S_1$ such that $f^{-1} \circ f$ is the identity map on S_1 and $f \circ f^{-1}$ is the identity map on $f(S_1)$. A map f has an inverse if and only if f is one-to-one.

The following basic result relates continuity as defined here for topological spaces with continuity as defined in *The General Derivative* for real vector spaces.

Let $f : U \subseteq \mathbf{R}^m \to \mathbf{R}^n$ be a mapping of topological spaces, where U has the subset topology induced by the Euclidean topology on $\mathbf{R}^m$, and $\mathbf{R}^n$ has the Euclidean topology. Then f is continuous on U as a map of topological spaces if and only if it is continuous on U in the sense of real and complex analysis, i.e., for any $p \in U$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(p, \delta)) \subseteq B(f(p), \varepsilon)$.

Proof: Assume that f is continuous on U in the sense of topological spaces. Then for any $p \in U$ and any ε, $B(f(p), \varepsilon)$ is open in $\mathbf{R}^n$, so $V = f^{-1}(B(f(p), \varepsilon))$ is open in U and therefore open in $\mathbf{R}^m$. This means we can find an open ball $B(p, \delta) \subseteq V$. Then $f(B(p, \delta)) \subseteq B(f(p), \varepsilon)$, as required.

Now assume that f is continuous on U in the sense of real analysis, and let $V \subseteq f(U)$ be an open set. We must show that $f^{-1}(V)$ is open in U. Choose any point $q \in V$ and any point $p \in f^{-1}(V)$ such that $f(p) = q$. Because V is open, there exists an open ball $B(q, \varepsilon) \subseteq V$. By continuity, there exists an open ball $B(p, \delta)$ such that $f(B(p, \delta)) \subseteq B(q, \varepsilon) \subseteq V$. But then $B(p, \delta) \subseteq f^{-1}(V)$. Since $B(p, \delta)$ is open and contains p and lies inside $f^{-1}(V)$, and this is true for every $p \in f^{-1}(V)$, $f^{-1}(V)$ is a union of open sets, so it is open. □

Homeomorphisms: Let T_1 and T_2 be topological spaces. A map $f : T_1 \to T_2$ is a **homeomorphism** if it is continuous and has a continuous inverse. Note the following:

1. The subset topology (defined above) lets us make any subset U of T_1 into a topological space. Therefore the definition of a homeomorphism applies to any map $f : U \subseteq T_1 \to T_2$.
2. A biholomorphic function (defined in § 1) is a homeomorphism from a subset $U \subseteq \mathbf{C}$ to $f(U) \subseteq \mathbf{C}$.
3. By the same argument made in § 1 for biholomorphic functions, a homeomorphism is an open map.

If a homeomorphism $f : T_1 \to T_2$ exists, then we say that the topological spaces T_1 and T_2 are **homeomorphic**. We say that subsets $U_1 \subseteq T_1$ and $U_2 \subseteq T_2$ are homeomorphic if they are homeomorphic as subsets of T_1 and T_2 respectively, with the induced topologies.

3.2. Properties

We now state some basic properties of topological spaces and maps between them.

Bounded subsets of $\mathbf{R}^n$: Let U be a subset of $\mathbf{R}^n$. We say that U is **bounded** if there exists a real number $r > 0$ such that $U \subseteq B(0, r)$. For example:

1. The unit circle is a bounded subset of $\mathbf{R}^2$.
2. The x axis is not a bounded subset of $\mathbf{R}^2$.

The following result is called the **Heine-Borel theorem**:

> *Let U be a subset of $\mathbf{R}^n$, $n > 0$, with the Euclidean topology. Then U is compact if and only if it is closed and bounded.*

You can find the proof of this theorem in any standard textbook on point-set topology, e.g., [Gaal 2009].[5]

Continuous images of compact sets: Let T_1 be a topological space, and let U be a subset of T. A **continuous image** of U is the image $f(U)$ of a continuous map $f: T_1 \to T_2$, where T_2 is another topological space.

> *The continuous image of a compact set is compact.*

Proof: Let T_1 and T_2 be topological spaces, let $f: T_1 \to T_2$ be a continuous map, and let $U \subseteq T_1$ be a compact set. Let $\{V_i\}_{i \in I}$ be an open cover of $f(U)$, not necessarily finite. Then because f is continuous, $\{f^{-1}(V_i)\}_{i \in I}$ is an open cover of U. Choose a finite subcover $\{f^{-1}(V_i)\}_{i \in J}$. Then $\{V_i\}_{i \in J}$ is a finite subcover of $f(U)$. □

Topological dimension: Let T be a topological space and $n > 0$ be a natural number. We say that T has **topological dimension** n if T is covered by a collection of open subsets, each of which is homeomorphic to an open subset of $\mathbf{R}^n$. Note that under this definition, not every topological space has a topological dimension. For example, the union of a line segment and a disc in $\mathbf{R}^2$, with the subset topology, has no topological dimension. More general definitions of topological dimension are possible, but this definition is simple and intuitive, and it is sufficient for our purposes.

The following statement asserts that, when it exists, the concept of topological dimension is well-defined:

> *Let T be a topological space. If T has topological dimension n, then it does not have topological dimension m, for any $m \neq n$.*

We prove the following special case. The general proof is similar.

> $\mathbf{R}$ *and* $\mathbf{R}^2$ *have distinct topological dimensions.*

Proof: It suffices to show that for any open set $U \subseteq \mathbf{R}^2$ and any continuous map $f: U \to \mathbf{R}$, f is not one-to-one. Choose an open ball $B(p, r)$ contained in U. Inside this open ball, put a cross $C = H \cup V$ consisting of two perpendicular line segments of equal length, a horizontal segment H parallel to the x axis and a vertical segment V parallel to the y axis, each with its midpoint at p, so that p is at the center of the cross. Let $q = f(p)$. Choose a point p_1 to the right of p on H, and let $q_1 = f(p_1)$. By continuity, we can choose a point $p_2 \neq p$ on V such that $q \leq q_2 = f(p_2) < q_1$. By continuity again, we can choose a point p_3 on H such that $q \leq q_3 = f(p_3) < q_2$.

The map f restricted to the subset topology on H is a continuous map from $H \subseteq \mathbf{R}$ to $\mathbf{R}$. By the intermediate value theorem from calculus, there exists a point p_4 between p_1 and p_3 on H such that $f(p_4) = q_2$. But then f maps both p_2 and p_4 to q_2, and $p_2 \neq p_4$, so f is not one-to-one. □

It is clear that $\mathbf{R}$ has topological dimension 1, and $\mathbf{R}^2$ has topological dimension 2.

3.3. Topological Surfaces

Let T be a topological space. We say that T is a **topological surface** if it has topological dimension 2. The following statements are clear:

1. $\mathbf{R}^2$ is a topological surface.
2. Any open subset of $\mathbf{R}^2$, with the subset topology, is a topological surface.

The corresponding statements for $\mathbf{C}$ are also clear.

4. Charts on Topological Surfaces

We now generalize the concepts presented in § 2 for biholomorphic charts on $\mathbf{C}$ to the case of complex charts on a topological surface.

4.1. Basic Definitions

Complex charts: On a topological surface T, a **complex chart** (chart for short, when there is no ambiguity) is a pair $C = (U, \phi)$ consisting of an open set $U \subseteq \mathbf{C}$ and a homeomorphism $\phi: U \to \mathbf{C}$. As before, we call U the **chart domain** associated with C. We call ϕ the **chart map** associated with C. We call $\phi(U)$ the **chart image** associated with C.

[5] In mathematics, **point-set topology** refers to the study of topological spaces and maps between them. **Algebraic topology** refers to the study of the algebraic properties of these concepts.

Let $a \in U$ be a complex number. If $\phi(a) = 0$, then we say that the chart (U, ϕ) is **centered at** a.

We may think of a complex chart as providing a complex coordinate on the chart domain. Note that a biholomorphic chart as defined in § 2.1 is a special case of this definition, where T is an open subset of **C**, and the homeomorphism ϕ is biholomorphic. In the case of a general topological surface T, the derivative of ϕ is not defined.

Transition functions: Let T be a topological surface, and let $C_i = (U_i, \phi_i)$ and $C_j = (U_j, \phi_j)$ be charts on T, where i and j are members of an index set I. We define the **transition function** $\phi_{ij} = \phi_j \circ \phi_i^{-1}$. This function is defined on $\phi_i(U_i \cap U_j)$, and its image is $\phi_j(U_i \cap U_j)$. It translates the chart image of C_i to the chart image of C_j on the area of overlap between the charts. The picture is the same one given in Figure 2 for biholomorphic charts on **C**.

Compatible charts: Again let T be a topological surface, and let $C_i = (U_i, \phi_i)$ and $C_j = (U_j, \phi_j)$ be charts on T, where i and j are members of an index set I. We say that the charts C_i and C_j are **compatible** if the transition functions ϕ_{ij} and ϕ_{ji} are biholomorphic. Note that in the case of biholomorphic charts (§ 2.1), all pairs of charts are compatible, because all chart maps are biholomorphic.

Complex atlases: A **complex atlas** on a topological surface T (atlas for short, when there is no ambiguity) is a family of charts $A = \{C_i = (U_i, \phi_i)\}_{i \in I}$, where I is an index set, the sets U_i cover T, and every pair of charts C_i and C_j in A is compatible. The compatibility condition is new; we require compatibility because the chart maps are not biholomorphic in general.

Let A_1 and A_2 be complex atlases. As before, we say that A_1 is **included in** A_2 if every chart of A_1 is a chart of A_2. In this case we write $A_1 \subseteq A_2$.

Compatible atlases: Fix a topological surface T, and let A_1 and A_2 be atlases on T. We say that A_1 and A_2 are **compatible** if every chart of A_1 is compatible with every chart of A_2 and vice versa. Compatibility induces an equivalence relation on the set of atlases of T, in which two atlases are equivalent if they are compatible.

In the case of biholomorphic charts (§ 2.1), every chart is compatible with every other chart. So all atlases on **C** are equivalent.

Complex structures: Fix a topological surface T. An equivalence class of atlases on T is called a **complex structure**. Every atlas is a member of one and only one equivalence class; therefore we may specify a complex structure by giving any atlas in the structure. In the case of biholomorphic charts, all atlases are equivalent, so there is only one structure, the set of all atlases.

The maximal atlas: Every complex structure has a maximal atlas, i.e., an atlas that contains every chart of every atlas in the structure. Thus we may specify a complex structure by giving a maximal atlas. In the case of biholomorphic charts, the maximal atlas is the set of all charts.

Charts on subsets: Let T be a topological space and $U \subseteq T$ be a subset. Then the subset topology (§ 3.1) makes U into a topological space, so we may use the definitions above to put charts and atlases on U.

Riemann surfaces: A **Riemann surface** is a pair $R = (T, A)$, where T is a connected topological surface whose topology is Hausdorff and second countable, and A is a maximal atlas on T. It is clear that **C** with the Euclidean topology and all biholomorphic charts is a Riemann surface, as is any connected open subset of **C** with all biholomorphic charts.

4.2. Functions and Maps

For the reasons discussed in § 2.2, we focus on holomorphic functions.

Functions: Let $R = (T, A)$ be a Riemann surface with maximal atlas $A = \{C_i\}$.

A **function** on R is a mapping $f : T \to \mathbf{C}$. For each chart $C_i = (U_i, \phi_i)$, we define the **local function** f_i as in § 2.2, i.e.,

$$f_i = (\phi_i^{-1})^* f = f \circ \phi_i^{-1}.$$

Note that f_i is a complex function from $\phi_i(U_i) \subseteq \mathbf{C}$ to **C**.

The same arguments given in § 2.2 go through to establish a one-to-one correspondence between complex functions f on R and families of local functions $\{f_i\}$ on the charts of A that satisfy property (1), i.e,

$$f_i = \phi_{ij}^* f_j.$$

Let f be a complex function on R. We say that f is **holomorphic** if every local function f_i on A is holomorphic.

Maps: Let $R_1 = (T_1, A_1)$ and $R_2 = (T_2, A_2)$ be Riemann surfaces.

A **map** f from R_1 to R_2 is a map $f: T_1 \to T_2$. As in § 2.2, for each pair (C_i, C_j), where C_i is a chart of A_1 and C_j is a chart of A_2, we define

$$f_{ij} = \phi_j \circ f_i = \phi_j \circ f \circ \phi_i^{-1}.$$

f_{ij} is a function from $V \subseteq \mathbf{C}$ to $\mathbf{C}$, where $V = \phi_i(U_i \cap f^{-1}(U_j))$. Its image is $\phi_j(f(U_i \cap f^{-1}(U_j))) \subseteq \mathbf{C}$.

Let $f: R_1 \to R_2$ be a map. We say that f is **holomorphic** if for every i and j, f_{ij} is holomorphic.

4.3. Differential Forms

For the reasons discussed in § 2.2, we focus on holomorphic functions and one forms.

Local zero forms: We define local zero forms, local points, and integration of local zero forms over local points as in § 2.3, after replacing the open set $V \subseteq \mathbf{C}$ and atlas A in the definitions with a Riemann surface $R = (T, A)$. The definitions depend only on the complex charts U_i, and not on V or R.

Zero forms: We define zero forms, points, and integration of zero forms over points as in § 2.3, after replacing the open set $V \subseteq \mathbf{C}$ and atlas A in the definitions with a Riemann surface $R = (T, A)$. Nothing in the definitions depends on the complex numbers in the open set V. The same arguments go through to show that the integral is independent of the choice of chart.

Local one forms: We define local one forms, local paths, and integration of local one forms over local paths as in § 2.3, after replacing the open set $V \subseteq \mathbf{C}$ and atlas A in the definitions with a Riemann surface $R = (T, A)$. Again the definitions depend only on the complex charts U_i, and not on V or R.

One forms: We define one forms, paths, and integration of one forms over paths as in § 2.3, after replacing the open set $V \subseteq \mathbf{C}$ and atlas A in the definitions with a Riemann surface $R = (T, A)$. For each i and j, ϕ_{ij}^* is biholomorphic, so condition (2), i.e., $\omega_i = \phi_{ij}^* \omega_j$, is well-defined. The same arguments go through to show that the integral is independent of the choice of chart.

Global one forms: Let $R = (T, A)$ be a Riemann surface. There is no direct analog to the concept of a differential form $f\ dz$ on T, because in general there is no complex identity function $z: T \to \mathbf{C}$. However, we do have the concept of a holomorphic function $f: R \to \mathbf{C}$ (§ 4.2). Further, for any chart C_i in A, the local function $f_i = (\phi_i^{-1})^* f$ is holomorphic. Therefore, given a holomorphic function f on R, we can define the following **global differential** df on R:

$$df = \{df_i\}_{i \in I}, \tag{3}$$

where df_i means $d(f_i)$, i.e., the derivative of the holomorphic function f_i. For all i and j,

$$\phi_{ij}^* df_j = d(\phi_{ij}^* f_j) = df_i,$$

so (3) yields a valid one form on R.

Note that we are slightly abusing notation here. The notation df does *not* mean the derivative of $f: R \to \mathbf{C}$; in general there is no such derivative. Instead, we are proceeding by analogy. In the case of a biholomorphic atlas on an open set V and a holomorphic function $f: V \to \mathbf{C}$, by the definition of the local one form ω_i with $\omega = df$, we have

$$(df)_i = (\phi_i^{-1})^* df = d((\phi_i^{-1})^* f) = d(f_i). \tag{4}$$

In the case of a Riemann surface R and a holomorphic function $f: R \to \mathbf{C}$, df is not a priori defined. However, we can assert that $(df)_i = d(f_i)$, as in (4), and then we can define df as shown in (3).

We can extend this notation as follows. Let f and g be holomorphic functions on R. In the case of biholomorphic charts, we have

$$(\phi_i^{-1})^*(f\ dg) = ((\phi_i^{-1})^* f)((\phi_i^{-1})^* dg) = f_i\ dg_i.$$

Therefore on a general Riemann surface, we define

$$f\ dg = \{f_i\ dg_i\}_{i \in I}. \tag{5}$$

For all i and j,

$$\phi_{ij}^*(f_j\ dg_j) = (\phi_{ij}^* f_j)\ (\phi_{ij}^* dg_j) = f_i\ d(\phi_{ij}^* g_j) = f_i\ dg_i,$$

so (5) yields a valid one form on R. This is the closest analog on R to a global one form $f\ dz$ on an open set $V \subseteq \mathbf{C}$.

5. The Riemann Sphere

We now present a fundamental example of a Riemann surface — the Riemann sphere.

5.1. The One-Point Compactification of R

We begin with a basic construction in one real dimension: adding a single point to $\mathbf{R}$ to form a compact topological space. Let C be the unit circle centered at the origin.

The punctured unit circle: We define a **punctured circle** to be a circle with one point deleted. Let $C^* = C - \{(0,1)\}$ be the punctured circle formed by deleting the point $(0,1)$ from C. Each point on C^* corresponds to exactly one angle in the interval $I = (-3\pi/2, \pi/2)$, via the mapping $\alpha(\theta)\colon I \to C^*$ given by $\alpha(\theta) = (\cos\theta, \sin\theta)$. We construct a mapping $\psi(\theta)\colon I \to \mathbf{R}$ as follows: for each angle θ in I, let $p = \alpha(\theta)$, let L be the line passing through $(0,1)$ and p, and let $\psi(\theta)$ be the intersection of L with the x axis.

Figure 4 shows how to construct $q = \psi(\theta)$ in the case where θ lies in the range $[-\pi/2, 0]$, and the x coordinate of q lies between 0 and the x coordinate of p. As shown, the right triangle with vertical and horizontal sides and with hypotenuse from $(0,0)$ to p has a vertical side of length $-\sin\theta$ and a horizontal side of length $\cos\theta$. Therefore the slope of L is $m = -(1-\sin\theta)/\cos\theta$, and the x coordinate of q is $-1 \cdot (1/m) = \cos\theta/(1-\sin\theta)$.

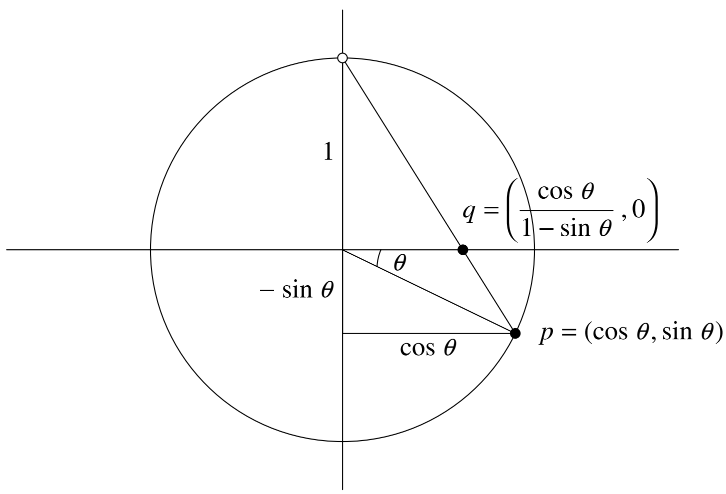

Figure 4: Projection from $(0,1)$, for $-\pi/2 \le \theta \le 0$.

Figure 5 shows how to construct $q = \psi(\theta)$ in the case where θ lies in the range $[0, \pi/2)$, and the x coordinate of p lies between 0 and the x coordinate of q. As shown, the right triangle with vertical and horizontal sides and with hypotenuse from $(0,1)$ to p has a vertical side of length $1-\sin\theta$ and a horizontal side of length $\cos\theta$. Therefore the slope of L is $m = -(1-\sin\theta)/\cos\theta$, and the x coordinate of q once again is $-1 \cdot (1/m) = \cos\theta/(1-\sin\theta)$.

These observations show that when θ lies in the range $[-\pi/2, \pi/2)$, we have

$$\psi(\theta) = \frac{\cos\theta}{1-\sin\theta}\,.$$

By symmetry, this mapping is valid for all θ in I.

Taking the derivative yields $\psi'(\theta) = 1/(1-\sin\theta)$. When θ lies in the interval $[-\pi/2, \pi/2)$, $\psi'(\theta) > 0$, and $\psi'(\theta)$ increases without bound as θ approaches $\pi/2$. Therefore the x coordinate of q strictly increases and becomes arbitrarily large as θ increases in the interval $[-\pi/2, \pi/2)$. Further, because $\psi(-\pi/2) = 0$, ψ maps every point on the unit circle in the interval $[-\pi/2, \pi/2)$ to a point on the nonnegative x axis, and this mapping is one-to-one. By symmetry, ψ is a one-to-one mapping from I to the x axis.

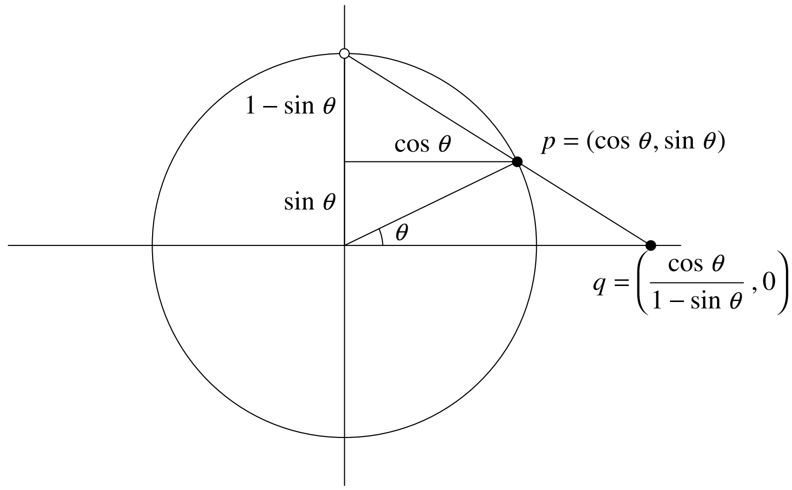

Figure 5: Projection from $(0, 1)$, for $0 \le \theta < \pi/2$.

Since the mapping $\alpha: I \to C^*$ is also one-to-one, ψ induces a one-to-one mapping $\phi = \psi \circ \alpha^{-1}: C^* \to \mathbf{R}$. The map ϕ is called the **projection** of points on C^* from $(0, 1)$ to the x axis.

Now put the subset topology on C^*, via its embedding in $\mathbf{R}^2$. A basis for this topology is given by the set of images under α of open intervals of angles, and each such image maps to an open interval in $\mathbf{R}$ under ϕ. Therefore ϕ is a homeomorphism from C^* to $\mathbf{R}$.

We have shown that $\mathbf{R}$, which is not bounded in $\mathbf{R}^2$, is homeomorphic to C^*, which is bounded in $\mathbf{R}^2$. This may seem counterintuitive. However, note that neither $\mathbf{R}$ nor C^* is closed in $\mathbf{R}^2$, so this result is consistent with the Heine-Borel theorem.

The unit circle: Put the subset topology on C, via its embedding in $\mathbf{R}^2$, and observe the following:

1. C contains C^* as a subset.
2. The subset topology on C^* as a subset of C is the same as the subset topology on C^* as a subset of $\mathbf{R}^2$.

Therefore, up to homeomorphism, we may think of C as a topological space consisting of $\mathbf{R}$ plus the point $(0, 1)$, which we call the **point at infinity** and denote ∞. An open set in C is either an open set in $C^* = \mathbf{R}$ or an open set containing $(0, 1) = \infty$. We may represent an open set U containing ∞ as $\{U - \infty\} \cup B$, where B is an open ball centered at ∞. Every such open ball B corresponds to the image under $\theta \mapsto (\cos\theta, \sin\theta)$ of a range of angles $(\pi/2 + \delta, \pi/2 - \delta)$, for some real number $\delta > 0$. Fix such an open ball B. Delete the point ∞ from B to form B^*, and form the projection $\phi(B^*)$ onto the real line. Let r be the projection from the point corresponding to $\pi/2 - \delta$. Then

$$\phi(B^*) = \{x \in \mathbf{R}: |x| > r\}.$$

C is a closed and bounded subset of $\mathbf{R}^2$, so by the Heine-Borel theorem it is compact. We call C with its subset topology and with the projection function $\phi: C^* \to \mathbf{R}$ the **one-point compactification** of $\mathbf{R}$.

5.2. The One-Point Compactification of C

Now we describe the analogous construction for the complex plane $\mathbf{C}$. Identify $\mathbf{C}$ with the xy plane in $\mathbf{R}^3$, and let S be the surface of the unit sphere centered at the origin in $\mathbf{R}^3$.

The punctured surface of the unit sphere: We define a **punctured spherical surface** to be a spherical surface with one point deleted. Let $S^* = S - \{(0, 0, 1)\}$ be the punctured spherical surface formed by deleting the point $(0, 0, 1)$ from S. Each point on S^* corresponds to exactly one pair of angles $\theta = (\theta_1, \theta_2)$ in the product of intervals $I = I_1 \times I_2 = [0, 2\pi) \times [-\pi/2, \pi/2)$, via the mapping $\alpha(\theta): I \to S^*$ given by $\alpha(\theta) = (\cos\theta_1, \sin\theta_1, \sin\theta_2)$. θ_1 is an angle in the y direction from the x axis, and θ_2 is an angle in the z direction from the line segment connecting the origin to $(\cos\theta_1, \sin\theta_1, 0)$.

We construct a mapping $\psi(\theta)\colon I \to \mathbf{C}$ as follows: for each pair of angles θ in I, let $p = \alpha(\theta)$, let L be the line passing through $(0, 0, 1)$ and p, and let $\psi(\theta)$ be the intersection of L with the xy plane. Figures 4 and 5 show the picture in the vertical plane through the origin and p. In polar coordinates, $\psi(\theta)$ is the point $re^{i\theta_1}$, where $r = \cos\theta_2/(1 - \sin\theta_2)$. The projection mapping $\phi = \psi \circ \alpha^{-1}$ is one-to-one from S^* to $\mathbf{C}$. By the same argument given in § 5.1, S^* is homeomorphic to $\mathbf{C}$.

The surface of the unit sphere: By the same argument given in § 5.1, we may think of S as a topological space consisting of $\mathbf{C}$ plus the point $(0, 0, 1)$, which we call the **point at infinity** and denote ∞. An open ball B centered at ∞ corresponds to all pairs of angles (θ_1, θ_2) such that $\pi/2 - \theta_2 < \delta$, for some $\delta > 0$. The projection $\phi(B^*)$ is given by

$$\{z \in \mathbf{C} \colon |z| > r\},$$

for some $r > 0$.

By Heine-Borel, S is compact. S with its subset topology and with the projection function $\phi\colon S^* \to \mathbf{C}$ is the **one-point compactification** of $\mathbf{C}$.

5.3. Projection from the Opposite Point

Now we investigate what happens when we project from the point opposite $(0, 1)$ on the unit circle or $(0, 0, 1)$ on the unit spherical surface. As before, let C be the unit circle centered at the origin in $\mathbf{R}^2$, and let S be the surface of the unit sphere centered at the origin in $\mathbf{R}^3$.

The unit circle: Let $p_1 = \infty = (0, 1)$, let $p_2 = (0, -1)$, and let $C_i^* = C - \{p_i\}$ for $i \in \{1, 2\}$. Let $\phi_1\colon C_1^* \to \mathbf{R}$ be the projection from p_1 described in § 5.1. Let ϕ_2 be the corresponding projection from p_2. Figure 6 shows the projection $\phi_2(p)$ of a point $p = \alpha(\theta)$ in C_2^* for θ in the range $[-\pi/2, 0]$. By an argument similar to the one we made in § 5.1, the x coordinate of q is $\cos\theta/(1 + \sin\theta)$, and the map $\phi_2\colon C_2^* \to \mathbf{R}$ is one-to-one. Further, we have the following facts:

1. $\phi_2(\infty) = 0$
2. Let θ be the angle of a point $p = \alpha(\theta)$ on $C_1^* \cap C_2^*$. Let $\psi_i(\theta)$ be the x coordinate of the projection from p_i through p. Then

$$\psi_1(\theta)\psi_2(\theta) = \left(\frac{\cos\theta}{1 - \sin\theta}\right)\left(\frac{\cos\theta}{1 + \sin\theta}\right) = \frac{\cos^2\theta}{1 - \sin^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} = 1,$$

   where the last step is valid because $\cos\theta \neq 0$. Thus $\phi_1(p) = 1/\phi_2(p)$, and vice versa.
3. The line passing through p_1 and p meets the line passing through p_2 and p at right angles. See the dashed line in Figure 6.

The surface of the unit sphere: Let $p_1 = \infty = (0, 0, 1)$, let $p_2 = (0, 0, -1)$, and let $S_i^* = S - \{p_i\}$ for $i \in \{1, 2\}$. Let $\phi_1\colon S_1^* \to \mathbf{C}$ be the projection from p_1 described in § 5.2. Let $\phi_2\colon S_2^* \to \mathbf{C}$ be the corresponding map obtained by projecting from p_2, where we reverse the sense of angles in the xy plane, i.e., we treat an angle from the positive x axis as positive if it is counterclockwise when looking up at the xy plane from p_2.

As before, $\phi_2(\infty) = 0$. Let p be a point in $S_1^* \cap S_2^*$, let $\phi_1(p) = r_1 e^{i\theta_1}$, and let $\phi_2(p) = r_2 e^{i\theta_2}$. By the argument in § 5.2, the value of r_1 is given by the geometry described in § 5.1 for the unit circle. Therefore, by the argument given above for ϕ_2 on the unit circle, $r_2 = 1/r_1$. Because we reversed the sense of angles in the xy plane, $\theta_2 = -\theta_1$. Therefore, as in the case of the unit circle, we have

$$\phi_2(p) = (1/r_1)e^{-i\theta_1} = 1/\phi_1(p).$$

5.4. Complex Charts

For each $i \in \{1, 2\}$, $\phi_i\colon S_i^* \to \mathbf{C}$ is a homeomorphism. Therefore, each pair $C_i = (S_i^*, \phi_i)$ is a chart on S. Moreover, the charts C_i cover S, and the transition functions are $\phi_{12}(z) = \phi_{21} = 1/z$, which is biholomorphic on $S_1^* \cap S_2^*$. Therefore $\{C_i\}_{i \in \{1,2\}}$ is an atlas on S; it has an associated maximal atlas A consisting of all charts on S compatible with each C_i. The pair $\mathbf{C}_\infty = (S, A)$ is a Riemann surface called the **Riemann sphere**.

The following result provides a natural way to think of a meromorphic function as a function that takes its values on the Riemann sphere.

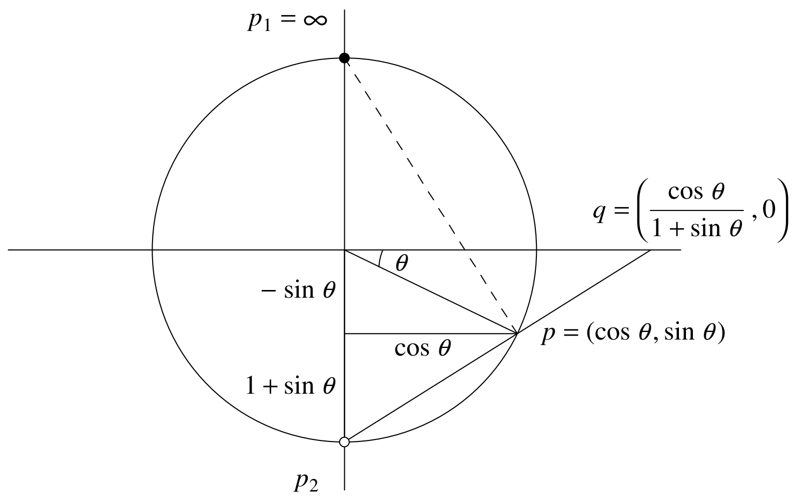

Figure 6: Projection from $p_2 = (0, -1)$, for $-\pi/2 \leq \theta \leq 0$.

Let $f: V - P \to \mathbf{C}$ be a meromorphic function, where $V \subseteq \mathbf{C}$ is a connected open set, and $P \subseteq V$ is a discrete set of poles of f. Let A be the maximal atlas consisting of all biholomorphic charts on V, and let R be the Riemann surface (V, A). Define a map $g: R \to \mathbf{C}_\infty$ as follows:

1. *For all $a \in V - P$, $g(a) = \phi_1^{-1}(f(a))$.*
2. *For all $a \in P$, $g(a) = (0, 0, 1) = \infty$.*

Then g is a holomorphic map between Riemann surfaces, as defined in § 4.2.

Proof: Let (U_i, ϕ_i) be a chart on V, let (U_j, ϕ_j) be a chart on S, and consider the function

$$g_{ij} = \phi_j \circ g \circ \phi_i^{-1}.$$

Let $b \in V$ be a point where g_{ij} is defined, and let $a = \phi_i^{-1}(b)$.

1. If $a \notin P$, then at b we have

$$g_{ij} = \phi_j \circ \phi_1^{-1} \circ f \circ \phi_i^{-1}.$$

By the definition of the Riemann surface R, ϕ_i^{-1} is holomorphic at b. By assumption, f is holomorphic at $a = \phi_1^{-1}(b) \notin P$. By the definition of the Riemann surface $\mathbf{C}_\infty$, ϕ_j must be compatible with ϕ_1, so $\phi_{1j} = \phi_j \circ \phi_1^{-1}$ is holomorphic at $f(a)$. Therefore g_{ij} is holomorphic at b.

2. Otherwise $a \in P$, and ϕ_2 is defined at $g(a) = \infty$, so at b we have

$$g_{ij} = \phi_j \circ \phi_2^{-1} \circ \phi_2 \circ g \circ \phi_i^{-1}.$$

By compatibility, $\phi_j \circ \phi_2^{-1}$ and ϕ_i^{-1} are holomorphic. Therefore it suffices to show that $\phi_2 \circ g$ is holomorphic at a. Because the zeros and poles of f are discrete,[6] there exists a neighborhood W of a such that on $W - \{a\}$ $h = \phi_2 \circ g$ is defined, and

$$h(z) = (\phi_2 \circ \phi_1^{-1} \circ f)(z) = 1/f(z).$$

Further, the limit of $h(z)$ as z approaches a is 0. Therefore by the theorem on removable singularities in the complex plane (*Calculus over the Complex Numbers*, § 5.2), we can extend h to a holomorphic function on W by setting $h(a) = 0$. But this extended function is exactly how we have defined $\phi_2 \circ g$ on W, because we have

$$\phi_2(g(a)) = \phi_2(\infty) = 0.$$

[6] The poles are discrete by the theory of Laurent series. See *Calculus over the Complex Numbers*, § 5.3. The zeros are discrete because the zeros of a nonconstant holomorphic function are isolated on a pathwise connected open set. See *Calculus over the Complex Numbers*, § 4.3. An open subset of $\mathbf{C}$ is connected if and only if it is pathwise connected. See [Lang 1999], III, Theorem 1.6.

□

Example 1: Let $V = \mathbf{C}$, let $P = \{0\}$, and let $f: V - P \to \mathbf{C}$ be the function $z \mapsto 1/z$, which is holomorphic on $\mathbf{C} - \{0\}$ and meromorphic on $\mathbf{C}$ with a pole at 0. The corresponding function $g: R \to \mathbf{C}_\infty$ maps z to $\phi_1^{-1}(1/z)$ for $z \in \mathbf{C} - \{0\}$ and maps 0 to ∞, and $\phi_2 \circ g$ is the identity function on all of $\mathbf{C}$.

Example 2: Let f and g be as in the previous example. We can construct other holomorphic functions by composing g with other charts on $\mathbf{C}_\infty$. For example, let B be an open ball in $\mathbf{C}$ that does not contain zero, let $C_j = \phi_2^{-1}(B)$, and let $\phi_j: C_j \to \mathbf{C}$ be the mapping $s \mapsto \phi_2(s)^2$. On B, we have $\phi_{2j}(z) = z^2$, and ϕ_{2j} is biholomorphic (see § 1). Therefore (C_j, ϕ_j) is a chart on $\mathbf{C}_\infty$. Further, on B we have

$$\phi_j \circ g = \phi_j \circ \phi_2^{-1} \circ \phi_2 \circ g = \phi_{2j} \circ id = z^2.$$

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