Definitions for Classical Algebraic Geometry

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This document defines concepts used in the area of mathematics known as **classical algebraic geometry**. This area, which flourished in the nineteenth century and reached its apex in the first half of the twentieth century, studies **affine and projective varieties**. Affine varieties are sets of solutions to equations of the form $p(x_1, \ldots, x_n) = 0$, where p is a polynomial in n variables over an algebraically closed field. Projective varieties are an extension of this concept. Classical algebraic geometry naturally extends the coordinate geometry that we all studied in high school, e.g., when drawing the graphs of curves such as $y = x^2$ or $x^2 - y = 0$ and intersecting these graphs with lines.

In contrast to the classical approach is **modern algebraic geometry** as developed by Alexander Grothendiek and others starting in the 1960s. Modern algebraic geometry studies **schemes**, which are an algebraic generalization of varieties. The modern, scheme-theoretic approach to algebraic geometry is extremely powerful, but it is also abstract and technical. Its primary tool is commutative algebra; it is farther removed from the "ordinary" geometry of vector spaces. Classical algebraic geometry provides the geometric intuition that motivates the more abstract modern approach. It is also important, e.g., in the study of complex algebraic curves, which are closely related to the study of Riemann surfaces.

This document assumes that you are familiar with the concepts discussed in my paper *Definitions for Commutative Algebra*. The motivation for this paper is the same as the motivation for that one: it is useful to collect definitions and basic results in one place for review and study.

Throughout this document, K refers to an algebraically closed field, and n refers to a natural number greater than zero.

1. Affine and Projective Space

In mathematics, a **space** is a set of points with some common property, e.g., a vector space or a topological space. There are two spaces associated with classical algebraic geometry: affine space and projective space. In this section, we define these spaces.

1.1. Affine Space

In the context of classical algebraic geometry, we refer to the vector space K^n as n-dimensional **affine space** and denote it $\mathbf{A}^n(K)$ or just $\mathbf{A}^n$, when the field K is implied. There are at least two reasons to define the term "affine space" instead of just saying K^n:

1. It provides a counterpart to projective space, which we define in the next section.
2. When K^n comes with a standard topology (e.g., in the case $K = \mathbf{C}$, which provides many of the classical examples in algebraic geometry), it lets us ignore that topology and use a different topology, which we define in § 5.

We write a specified point a of $\mathbf{A}^n$ as an n-tuple of coordinates: $a = (a_1, \ldots, a_n) = \{a_i\}$. We write a variable point as $z = (z_1, \ldots, z_n) = \{z_i\}$. Note that the coordinate indices go from 1 through n.

By tradition, $\mathbf{A}^1$ is called the **affine line**. $\mathbf{A}^2$ is called the **affine plane**.

1.2. Projective Space

Affine space corresponds well to our geometric intuition of points embedded in a vector space. However, affine space has the inconvenient property that a pair of lines may or may not intersect in a point. For example, the lines $z_1 = 0$ and $z_2 = 0$ in $\mathbf{A}^2$ intersect in the point $(0, 0)$, but the lines lines $z_1 = 0$ and $z_1 = 1$ in $\mathbf{A}^2$ do not intersect (i.e., they are parallel lines). This fact makes it inconvenient to count intersections. Therefore we introduce the concept of n-dimensional **projective space**, written $\mathbf{P}^n(K)$ or just $\mathbf{P}^n$, when the field K is implied. $\mathbf{P}^n$ is given by either of the following equivalent definitions:

1. It is the set of lines through the origin in K^{n+1}.
2. It is the set of equivalence classes of points $a \neq 0$ in K^{n+1}, under the relation $a \sim ka$ for any a in K^{n+1} and any $k \neq 0$ in K.

In the case $K = \mathbf{C}$, we may also think of $\mathbf{P}^n$ as the set of equivalence classes of points on the unit sphere C centered at the origin in $\mathbf{C}^{n+1}$, under the relation that identifies each point a with its antipodal point on C, i.e., the point $a' \neq a$ on C and on the line L containing 0 and a.

According to definition 2, an element a of $\mathbf{P}^n$ is an equivalence class consisting of all tuples $(ka_0, \ldots, ka_n) = \{ka_i\}$ for a fixed set of values $\{a_i\}$, not all zero, and all $k \neq 0$. If $\{a_i\}$ is any representative of the class a, then we write $[a_0, \ldots, a_n]$ or $[a_i]$ to denote the class, and we call the values a_i **homogeneous coordinates** of the element a. We write $Z = [Z_0, \ldots, Z_n] = [Z_i]$ to represent a variable point of $\mathbf{P}^n$, with the understanding that $[kZ_i]$ represents the same point, for any $k \neq 0$. Note that the coordinate indices go from zero through n.

By tradition, $\mathbf{P}^1$ is called the **projective line**. $\mathbf{P}^2$ is called the **projective plane**.

Fix a point a in $\mathbf{P}^n$, and fix an index $j \in [0, n]$.

1. The definitions imply that either $a_j = 0$ or $a_j \neq 0$ for all representatives of a.
2. In the second case, a has a representative $\{a_i/a_j\}_{i \in [0,n]}$, and this is the unique representative with 1 as its j coordinate. Moreover, the family of coordinates $\{a_i/a_j\}$ for $i \neq j$ defines an element of $\mathbf{A}^n$.
3. Let U_j be the set of elements $[a_j]$ of $\mathbf{P}^n$ with $a_j \neq 0$. By item 2, we may identify U_j with $\mathbf{A}^n$.

The family of sets $\{U_i\}$ defined above is called the **standard cover** of $\mathbf{P}^n$ by affine spaces $\mathbf{A}^n$. Since each element a in $\mathbf{P}^n$ has $a_i \neq 0$ for some i, the sets U_i cover $\mathbf{P}^n$. The projection of $\mathbf{P}^n$ onto $U_i = \mathbf{A}^n$ motivates the name "projective space."

Returning to the example of the lines $z_1 = 0$ and $z_1 = 1$ in $\mathbf{A}^2$:

1. A general point $[Z_0, Z_1, Z_2]$ in $U_0 \subseteq \mathbf{A}^2$ is represented by the tuple $(1, Z_1/Z_0, Z_2/Z_0)$ in $\mathbf{A}^3$ and corresponds to the point $(Z_1/Z_0, Z_2/Z_0)$ in $\mathbf{A}^2$.
2. Thus we may lift the lines to $\mathbf{P}^2$ by substituting $z_1 = Z_1/Z_0$ in the equations defining the lines. When we do this, the equations become $Z_1 = 0$ and $Z_1 = Z_0$. The projections of these lines in $\mathbf{P}^2$ onto U_0 are the original lines in $\mathbf{A}^2$.
3. Whereas the lines in $\mathbf{A}^2$ do not intersect, the corresponding lines in $\mathbf{P}^2$ intersect in the point $[0, 0, 1]$.

If we think of the space defined by $Z_0 = 1$ as a copy S of $\mathbf{A}^2$ in $\mathbf{P}^2$, then the complementary space $\mathbf{P}^2 - S$ defined by $Z_0 = 0$ is a copy of the projective line $\mathbf{P}^1$. Let $L = \mathbf{P}^1$. Then $\mathbf{P}^2 = S \cup L$. With respect to S, the line L is "at infinity," in the sense that parallel lines in S, when lifted to $\mathbf{P}^2$, intersect at a point on L.

More generally, when we set $Z_0 = 1$ in $\mathbf{P}^n$, the remaining $n-1$ coordinates define a point in $\mathbf{A}^n$. When we set $Z_0 = 0$, the remaining $n-1$ coordinates define a point in $\mathbf{P}^{n-1}$. Therefore we can think of $\mathbf{P}^n$ as a copy of $\mathbf{A}^n$ together with a copy of $\mathbf{P}^{n-1}$ "at infinity."

2. Affine and Projective Varieties

In this section we define the main objects of study in classical algebraic geometry: affine and projective varieties.

2.1. Affine Varieties

We write $K[z]$ to denote the polynomial ring $K[z_1, \ldots, z_n]$, and we write $p(z)$ to denote a polynomial $p(z_1, \ldots, z_n)$ in $K[z]$.

Let $p(z)$ be a polynomial in $K[z]$. Then p defines a function $p\colon \mathbf{A}^n \to K$ given by $a \mapsto p(a)$. The **zero set** or **zero locus** of p is the set of elements a in $\mathbf{A}^n$ such that $p(a) = 0$.

Let $F = \{p_\alpha\}$ be a family of polynomials in $K[z]$. The **zero set** or **zero locus** of F is the set of points a in $\mathbf{A}^n$ such that a is in the zero set of each p_α.

An **affine variety** is a set $V \subseteq \mathbf{A}^n$ such that V is the zero set of a family of polynomials $F = \{p_\alpha\}$ in $K[z]$. We write $V(F)$ to denote the zero set of F. We also say that $V(F)$ is the affine variety **generated by** or **cut out by** the polynomials F.

The empty set is an affine variety. For example, it is the zero set of the polynomial 1, where 1 is the multiplicative identity of K.

2.2. Projective Varieties

We write $K[Z]$ to denote the polynomial ring $K[Z_0, \ldots, Z_n]$, and we write $P(Z)$ to denote a polynomial $P(Z_0, \ldots, Z_n)$ in $K[Z]$. Recall that $P(Z)$ is **homogeneous** if every term in $P(Z)$ has d variable factors for some $d \geq 0$. The number d is called the **degree** of the homogeneous polynomial. For example, the polynomial $Z_1^2 - Z_0 Z_2$ is homogeneous of degree 2.

Let $P(Z)$ be a polynomial in $K[Z]$. In general, P does not define a function from $\mathbf{P}^n$ to K. However, if P is homogeneous, then the zero set of P is well defined, because the property $P(a) = 0$ does not depend on the choice of homogeneous coordinates. Indeed, if p has degree d, then we have $P(ka) = k^d P(a)$ for any $k \neq 0$, so $P(a) = 0$ if and only if $P(ka) = 0$. Similarly, the zero set of a family of homogeneous polynomials in $K[Z]$ is well defined.

A **projective variety** is a set $V \subseteq \mathbf{P}^n$ such that V is the zero set of a family F of homogeneous polynomials $F = \{P_\alpha\}$ in $K[Z]$. We write $V(F)$ to denote the zero set of F. We also say that $V(F)$ is the projective variety **generated by** or **cut out by** the polynomials F.

The empty set is a projective variety. For example, it is the zero set of the polynomials $\{Z_i = 0\}_{i \in [0,n]}$, since there is no point a of $\mathbf{P}^n$ such that all the homogeneous coordinates of a are zero.

2.3. The Relationship Between Affine and Projective Varieties

Let $V \subseteq \mathbf{A}^n$ be an affine variety. For each i, V is the intersection of $U_i \subseteq \mathbf{P}^n$ with a projective variety $W_i \subseteq \mathbf{P}^n$. Indeed, suppose that V is generated by the polynomials $\{p_\alpha\}$. For each α, construct a homogeneous polynomial $P_\alpha(Z)$ as follows:

1. Renumber the variables in p_α from 0 through n, skipping i.
2. Replace each variable z_j in the result of step 2 with Z_j/Z_i.
3. Multiply every term in the result of step 3 by the highest power of Z_i appearing in any term.

For example, let $p_\alpha(z) = z_1 + z_2^2$, and let $i = 1$. Then step 1 yields $z_0 + z_2^2$, step 2 yields $(Z_0/Z_1) + Z_2^2$, and step 3 yields $P_\alpha(Z) = Z_0 Z_1 + Z_2^2$. Now let W_i be the zero set in $\mathbf{P}^n$ of the polynomials $\{P_\alpha\}$. Then $V = U_i \cap W_i$.

Let $V \subseteq \mathbf{P}^n$ be a projective variety. The intersection of V with any of the sets $U_i \subseteq \mathbf{P}^n$ is an affine variety in $\mathbf{A}^n$. Indeed, if $P_\alpha(Z)$ is a homogeneous polynomial of degree d in $K[Z]$, then when $Z_i \neq 0$, the zero set of P_α equals the zero set of $P_\alpha(Z)/Z_i^d$; thus we may divide each of the d factors of each term of P_α by Z_i. After doing this we may replace Z_i/Z_i with 1, replace Z_j/Z_i with z_j, and renumber the variables z_j from 1 to n; then $V \cap U_i$ is the zero set of the resulting polynomials $p_\alpha(z)$.

2.4. Subvarieties; Irreducible Varieties

If V and W are affine varieties and $W \subseteq V$ as a set, then we say that W is a **subvariety** of V. The analogous definition holds for projective varieties V and W.

An affine variety V is **irreducible** if it is nonempty and it cannot be written as the union $V = V_1 \cup V_2$ for any pair V_1 and V_2 of affine varieties, neither of which is equal to V. The analogous definition holds for projective varieties. For example:

1. $\mathbf{A}^n$ is irreducible, because the only polynomial in $K[z]$ that is zero on all of $\mathbf{A}^n$ is the zero polynomial.
2. $\mathbf{P}^n$ is irreducible, because the only homogeneous polynomial in $K[Z]$ that is zero on all of $\mathbf{P}^n$ is the zero polynomial.

The terminology used here is consistent with, e.g., [Harris 1992]. Some sources (e.g., [Fulton 2008], [Hartshorne 1977]) require that an affine or projective variety be irreducible. In these sources, what we have defined as an affine variety is called an affine **algebraic set**; and similarly for projective varieties. I prefer the terminology used here because it is more general: definitions and results that pertain to varieties here also apply to algebraic sets. When we need a variety to be irreducible, we will say so explicitly.

3. Ideals Corresponding to Affine and Projective Varieties

In this section we explore the connection between affine and projective varieties and polynomial ideals.

3.1. Affine Varieties

Fix a set $S \subseteq \mathbf{A}^n$. We write $I(S)$ to denote the set of polynomials p in $K[z]$ that vanish on S, i.e., $p(a) = 0$ for all points a in S. Observe the following:

1. $I(S)$ is not empty, because the polynomial 0 vanishes everywhere on $\mathbf{A}^n$.
2. If p is a member of $I(S)$, then qp is a member of S for any q in $K[z]$, because for any a in S we have $(qp)(a) = q(a)p(a) = q(a) \cdot 0 = 0$.

Therefore $I(S)$ is an ideal of $K[z]$. We call $I(S)$ the **vanishing ideal** corresponding to the set S.

Fix an ideal I of $K[z]$. Recall the following definitions (see *Definitions for Commutative Algebra*, § 6):

1. The **radical** of I, written $\text{rad}(I)$, is the ideal of $K[z]$ consisting of all polynomials p such that $p^m \in I$ for some $m > 0$. For example, the radical of the principal ideal (x^2) is (x).
2. We say that I is **radical** if $\text{rad}(I) = I$. For example, (x) is radical, and (x^2) is not.

Fix a subset $S \subseteq \mathbf{A}^n$, a point a in S, and a polynomial $p \in I(S)$. Suppose that $p = q^n$ for some $n > 1$. Because $p(a) = 0$, and K is an integral domain, we must have $q(a) = 0$, i.e., $q \in I(S)$. Therefore $I(S)$ is a radical ideal.

Fix an ideal I of $K[z]$.

1. By definition, the zero set of the polynomials in I is an affine variety (§ 2). We denote this affine variety $V(I)$.
2. We have $I(V(I)) = \text{rad}(I)$. This statement is the affine version of the **Hilbert Nullstellensatz** (in English, "zero statement") (proof omitted). It establishes a one-to-one correspondence between affine varieties $V \subseteq \mathbf{A}^n$ and radical ideals $I \subseteq K[z]$.
3. $V(I)$ is irreducible if and only if I is a prime ideal (proof omitted).

Fix an ideal I of $K[z]$. Then I is finitely generated, i.e., there exists a finite family F of polynomials in $K[z]$ such F generates I. This is the **Hilbert basis theorem** (proof omitted).

Fix a family F of polynomials in $K[z]$, not necessarily finite. We write $I(F)$ to denote the ideal generated by F.

1. We have $V(I(F)) = V(F)$. Indeed, if a is in the zero set of all the polynomials in $I(F)$, then it is in the zero set of all the polynomials in $F \subseteq I(F)$. On the other hand, each element $I(F)$ is a polynomial $p = \sum_i q_i p_i$ with $p_i \in F$ for all i; so if a is in the zero set of all the p_i, then $p(a) = 0$.
2. By the Hilbert basis theorem, $V(I(F)) = V(I(F'))$, where F' is a finite family of polynomials in $K[z]$. By item 1, $V(F) = V(I(F)) = V(I(F')) = V(F')$. Therefore every affine variety is generated by a finite family of polynomials.

3.2. Projective Varieties

Fix an ideal I in $K[Z]$. If I is generated by a set of homogeneous polynomials, then we call I a **homogeneous ideal**.

Fix a set $S \subseteq \mathbf{P}^n$. We write $I(S)$ to denote the homogeneous ideal generated by the homogeneous polynomials P that vanish on S, i.e., $P(a) = 0$ for all a in S. We call $I(S)$ the **vanishing ideal** corresponding to S. By the same argument that we made in § 3.1 for affine varieties, $I(S)$ is a radical ideal.

Let I be a homogeneous ideal of $K[Z]$.

1. Let F be a family of homogeneous polynomials that generate I. By definition, the zero set of F is a projective variety (§ 2). We denote this projective variety $V(F)$.
2. Let F_1 and F_2 be two such generating families. Then each element of F_1 may be written as a sum of terms, each of which is a polynomial in $K[Z]$ times a polynomial in F_2. Therefore an element in the zero set of F_2 is in the zero set of F_1 and vice versa, so the set $V(F)$ is independent of the generators chosen. We write $V(I)$ to denote the projective variety $V(F)$ for any generators F of I.
3. We have $I(V(I)) = \text{rad}(I)$. This statement is the projective version of the **Hilbert Nullstellensatz** (proof omitted). It establishes a one-to-one correspondence between projective varieties $V \subseteq \mathbf{P}^n$ and radical

homogeneous ideals $I \subseteq K[Z]$.

4. $V(I)$ is irreducible if and only if I is a prime ideal (proof omitted).

By the Hilbert basis theorem, a homogeneous ideal I of $K[Z]$ is finitely generated.

By the same arguments we made in § 2.1 for affine varieties:

1. For any family F of homogeneous polynomials, $V(I(F)) = V(F)$, where $I(F)$ is the homogeneous ideal generated by F.
2. By the Hilbert basis theorem, every projective variety is generated by a finite family of homogeneous polynomials.

3.3. Irreducible Components

Any nonempty affine variety V may be uniquely expressed as a finite union of irreducible affine varieties $\{V_i\}$ with V_i not included in V_j for $i \neq j$; and similarly for projective varieties (proof omitted). This statement follows from a general theorem on the primary decomposition of ideals in Noetherian rings (see *Definitions for Commutative Algebra*, §§ 17 and 20). The varieties V_i are called the **irreducible components** of V.

For example, let $V \subseteq \mathbf{A}^2$ be the variety generated by $z_1 z_2$. It is a union of two lines. The irreducible components of V are the lines $z_1 = 0$ and $z_2 = 0$.

4. The Dimension of an Affine or Projective Variety; Curves and Surfaces

Due to the deep connection between the algebra of $K[z]$ and the geometry of affine varieties in $\mathbf{A}^n$, there are many equivalent definitions of the dimension of an affine variety; and similarly for projective varieties. Here we give the most elementary definition of the dimension of a variety; we will give other definitions in § 13, after we have developed the concepts necessary to state them.

4.1. Affine Varieties

Let $V \subseteq \mathbf{A}^n$ be a nonempty affine variety. We define the **dimension** of V, written dim V, to be the maximal length m over all chains

$$V_0 \subset V_1 \subset \cdots \subset V_m \subseteq V,$$

where each V_i is an irreducible affine variety, and $\subset$ denotes strict inclusion of sets. For example:

1. The dimension of a point is zero. For example, $\{(1, 1)\}$ is an irreducible affine subvariety of $\mathbf{A}^2$ of dimension zero: it is the zero set of the polynomials $z_1 - 1$ and $z_2 - 1$.
2. dim $\mathbf{A}^n = n$. Each of the subvarieties given by $z_i = 0$, for $i \in [1, n]$, is is a copy of $\mathbf{A}^{n-1}$. Each of these copies is an irreducible subvariety of dimension $n - 1$,
3. Let V be the zero set of $p(z) = z_1^2 - z_2$ in $\mathbf{A}^2$. Then V has dimension one. The irreducible affine subvarieties of dimension zero are the points of V.

4.2. Projective Varieties

Let $V \subseteq \mathbf{P}^n$ be a nonempty projective variety. We define the **dimension** of V, written dim V, to be the maximal length m over all chains

$$V_0 \subset V_1 \subset \cdots \subset V_m \subseteq V,$$

where each V_i is an irreducible projective variety, and $\subset$ denotes strict inclusion of sets. For example:

1. The dimension of a point is zero. For example, $\{[1, 1, 1]\}$ is an irreducible projective subvariety of $\mathbf{P}_2$ of dimension zero: it is the zero set of the polynomials $Z_1 - Z_0$ and $Z_2 - Z_0$.
2. dim $\mathbf{P}^n = n$. Each of the subvarieties given by $Z_i = 0$, for $i \in [1, n]$, is a copy of $\mathbf{P}^{n-1}$. Each of these copies is an irreducible subvariety of dimension $n - 1$,
3. Let V be the zero set of $P(Z) = Z_1^2 - Z_0 Z_2$ in $\mathbf{P}^2$. Then V has dimension one. The irreducible projective subvarieties of dimension zero are the points of V.

4.3. Irreducible Components

We define the dimension of the empty variety to be -1.

Let V be a nonempty affine or projective variety, and let $\{V_i\}$ be its irreducible components (§ 3.3).

1. From the definitions in §§ 4.1 and 4.2, it is clear that the dimension of V is the maximum of the dimensions of the V_i.
2. If all the V_i have the same dimension d, then we say that V has **pure dimension** d.
3. Let p be a point of V. We define the **local dimension** of V at p, written $\dim_p V$, to be the maximum of the dimensions of the irreducible components V_i that contain p.

4.4. Curves and Surfaces

By tradition, we associate the following terms with the dimension of an affine or projective variety V:

1. A variety V of pure dimension one (§ 4.3) is called a **curve**. If V is generated by linear polynomials, then we call it a **line**.
2. A curve in $\mathbf{A}^2$ is called an **affine plane curve**. A curve in $\mathbf{P}^2$ is called a **projective plane curve**.
3. A variety of pure dimension $n-1$ in $\mathbf{A}^n$ or in $\mathbf{P}^n$ is called a **hypersurface**. An affine or projective variety V is a hypersurface if and only if it is generated by a single polynomial (proof omitted). A hypersurface in $\mathbf{A}^3$ or in $\mathbf{P}^3$ has dimension two and is called a **surface**.
4. A hypersurface generated by a linear polynomial is called a **hyperplane**. For example, the set of points in $\mathbf{A}^n$ such that $z_1 = 0$ is a hyperplane. A hyperplane in $\mathbf{A}^3$ or in $\mathbf{P}^3$ has dimension two and is called a **plane**.
5. A d-dimensional variety generated by linear polynomials is called a **linear subspace** or a **d-plane**. Note that a plane in $\mathbf{A}^3$ or $\mathbf{P}^3$ is a 2-plane.

Curves in affine and projective space are often grouped together under the subject of **algebraic curves**. See, e.g., [Fulton 2008]. Algebraic curves over the complex numbers (i.e., for $K = \mathbf{C}$) are called **complex algebraic curves**. See, e.g., [Kirwan 1992]. Complex algebraic curves have complex dimension one but real dimension two, so they are sometimes treated as topological surfaces, e.g., in the study of Riemann surfaces.

5. The Zariski Topology

Recall that a **topological space** is a pair (S, O), where S is a set and O is a set of subsets of S satisfying the following axioms:

T1. The empty set and S are elements of O.

T2. Any union of elements of O is an element of O.

T3. Any intersection of finitely many elements of O is an element of O.

The set O is called a **topology** on S. The elements of O are called the **open sets** of the topology. A set $T \subseteq S$ is **closed** if its complement $S - T$ in S is open; the empty set and S are both closed (and both open). For more information on topological spaces, see § 23 of *Definitions for Commutative Algebra*.

In this section, we use the geometry of affine and projective varieties to define a topology called the **Zariski topology** on affine and projective space.

5.1. Affine Space

Let O be the set of subsets T of $\mathbf{A}^n$ such that the complement $\mathbf{A}^n - T$ is an affine variety. In other words, let O be the topology on $\mathbf{A}^n$ in which the closed sets are the affine varieties. Then O is a topology on $\mathbf{A}^n$:

T1. $\mathbf{A}^n$ is the affine algebraic variety generated by the zero ideal, and $\emptyset$ is the affine algebraic variety generated by the ideal $K[z]$. Therefore $\emptyset = \mathbf{A}^n - \mathbf{A}^n$ and $\mathbf{A}^n = \mathbf{A}^n - \emptyset$ are open sets.

T2. By taking complements, it suffices to show that any intersection of closed sets is a closed set. Let $\{V_i\}$ be a family of closed sets, i.e., affine varieties. Each V_i is generated by a family of polynomials F_i. Further, $\bigcap_i V_i = \bigcap_i V(F_i) = V(\bigcup_i F_i)$. Therefore $\bigcap_i V_i$ is an affine variety, i.e., closed.

T3. By taking complements, it suffices to show that any union of finitely many closed sets is a closed set. Then by induction, it suffices to show that the union $V_1 \cup V_2$ of any two affine varieties is an affine variety. Let $V_1 = V(F_1)$ and $V_2 = V(F_2)$ for families of polynomials $F_1 = \{p_\alpha\}$ and $F_2 = \{q_\beta\}$. Then the union $V_1 \cup V_2$ is the affine variety $V(F)$, where F is the set of all $p_\alpha q_\beta$ for p_α in F_1 and q_β in F_2. Indeed, if a point a is in the zero set of F_1 or of F_2, then $(p_\alpha q_\beta)(a) = 0$ for all α and β, so $a \in V(F)$. On the other hand, if a is outside the zero sets of both F_1 and F_2, then there exist p_α and q_β such that $p_\alpha(a) \neq 0$ and $q_\beta(a) \neq 0$; in this case $(p_\alpha q_\beta)(a) \neq 0$, so $a \notin V(F)$.

We call O the **Zariski topology** on $\mathbf{A}^n$.

Let (S, O) be a topological space. A **basis** for the topology O is a set B of subsets of S such that any open set may be written as a union of sets in B.

Denote by $V(p)$ the affine variety generated by the polynomial p. Because every affine variety is generated by a set of polynomials, every open set in the Zariski topology is given by a union of open sets $\mathbf{A}^n - V(p)$ for polynomials p. Therefore the open sets $\mathbf{A}^n - V(p)$, as p ranges over $K[z]$, form a basis for the Zariski topology. These sets are called the **distinguished open sets** of $\mathbf{A}^n$.

Fix a subset S of $\mathbf{A}^n$. We give S the **subset topology**, i.e., the topology in which a subset $T \subseteq S$ is open if and only if it is the intersection of S with an open set of $\mathbf{A}^n$. When a subset T of S is open (respectively closed) in the subset topology on S, we say that T is **open in** S (respectively **closed in** S).

Fix an affine variety V.

1. The **distinguished open sets** of V are the intersections of V with the distinguished open sets of $\mathbf{A}^n$, i.e., the sets $V - V(p)$ for polynomials p in $K[z]$. These sets are open in V.
2. A subset T of V is closed in V if and only if it is closed in $\mathbf{A}^n$, i.e., it is an affine variety. Indeed, fix a subset T of V. If T is closed in $\mathbf{A}^n$, then $V - T$ is the intersection of the open set $\mathbf{A}^n - T$ with V, so $V - T$ is open in V, so T is closed in V. On the other hand, if T is closed in V, then $V - T$ is open in V, so it is the intersection of V with an open set in $\mathbf{A}^n$. Therefore T is the intersection of V with a closed set in $\mathbf{A}^n$, i.e. an affine variety W. Thus T is the affine variety $V \cap W$.

Let (S, O) be a topological space, and fix a subset T of S.

1. The **closure** of T in S is the intersection of all the closed sets of S that contain T.
2. T is **dense in** S if its closure in S is all of S. T is dense in S if and only if it has a nonempty intersection with every nonempty open subset of S. Indeed, fix a nonempty open subset U of S. If T has empty intersection with U, then $S - U$ is a closed set containing T and not equal to S, so T is not dense in S. On the other hand, if T is not dense in S, then there exists a closed set U containing T and not equal to S, and $S - U$ is a nonempty open set that has empty intersection with T.

Let V be an irreducible affine variety.

1. For any two nonempty subsets T and U of V that are open in V, $T \cap U \neq \emptyset$. Indeed $W_1 = V - T$ and $W_2 = V - U$ are closed sets in V, so they are affine varieties. Further, $W_1 \cup W_2 = V - (T \cap U)$, so if $T \cap U = \emptyset$, then $V = W_1 \cup W_2$, and so V is not irreducible.
2. Every nonempty subset T of V that is open in V is dense in V. This statement follows from item 1 and from the fact, shown above, that T is dense in V if and only if it has nonempty intersection with every nonempty open subset U of V.

In particular, these facts hold when $V = \mathbf{A}^n$ and $T \subseteq \mathbf{A}^n$ is an open set.

5.2. Projective Space

Let O be the set of subsets T of $\mathbf{P}^n$ such that the complement $\mathbf{P}^n - T$ is a projective variety. In other words, let O be the topology on $\mathbf{P}^n$ in which the closed sets are the projective varieties. Then the same arguments that we made in § 5.1 for the Zariski topology on $\mathbf{A}^n$ establish that O is a topology on $\mathbf{P}^n$. We call O the **Zariski topology** on $\mathbf{P}^n$. All the definitions and facts stated in § 5.1 apply to the Zariski topology on $\mathbf{P}^n$, after replacing "affine" with "projective," $\mathbf{A}^n$ with $\mathbf{P}^n$, and polynomials $p(z)$ in $K[z]$ with homogeneous polynomials $P(Z)$ in $K[Z]$.

6. Quasi-Affine and Quasi-Projective Varieties; General Varieties

In this section we use the Zariski topology (§ 5) to generalize the concepts of affine and projective varieties.

6.1. Quasi-Affine Varieties

Fix an affine variety V. A subset $W \subseteq V$ that is open in V is called a **quasi-affine variety**.

1. Every affine variety V is a quasi-affine variety. Indeed, by the definition of the subset topology, V is open in V.
2. There exist quasi-affine varieties W such that W is not an affine variety. For example $X = \mathbf{A}^2 - \{(0,0)\}$ is a quasi-affine variety, because it is the complement of the zero set of the polynomials z_1 and z_2. However, X is not an affine variety (proof omitted).

Fix an affine variety V and a quasi-affine variety $W \subseteq V$. Then any set $U \subseteq W$ that is open in W is a quasi-affine variety. Indeed, since W is open in V, $W = V \cap X$ for some set X that is open in $\mathbf{A}^n$. Since U is open in W, U is the intersection of W and an open set of V, i.e., $U = W \cap (V \cap Y)$ for some set Y that is open in $\mathbf{A}^n$. Therefore

$$U = (V \cap X) \cap (V \cap Y) = V \cap (X \cap Y) = V \cap Z,$$

where $Z = X \cap Y$ is open in $\mathbf{A}^n$, so U is open in V. Thus any definition or result that is valid for a quasi-affine variety W is also valid for any set $U \subseteq W$ that is open in W.

6.2. Quasi-Projective Varieties

Fix a projective variety V. A subset $W \subseteq V$ that is open in V is called a **quasi-projective variety**.

1. Every affine variety $V \subseteq \mathbf{A}^n$ is a quasi-projective variety. Indeed, as we observed in § 2.3, V is the intersection of the set U_i with a projective variety $W_i \subseteq \mathbf{P}^n$. U_i is open in $\mathbf{P}^n$, because it is the complement of the zero locus of the polynomials $\{Z_j\}_{j \neq i}$. Therefore V is open in W_i.
2. Every projective variety V is a quasi-projective variety. Indeed, by the definition of the subset topology, V is open in V.
3. There exist quasi-projective varieties W such that W is neither an affine variety nor a projective variety. For example $X = \mathbf{A}^2 - \{(0,0)\}$ is a quasi-projective variety, because it is the intersection of U_0 and the complement of $\{[1,0,0]\}$ in $\mathbf{P}^2$. However, X is neither an affine variety nor a projective variety (proof omitted).

Fix a projective variety V and a quasi-projective variety $W \subseteq V$.

1. By the argument that we made in § 6.1 for quasi-affine varieties, any set $U \subseteq W$ that is open in W is a quasi-projective variety. Thus any definition or result that is valid for a quasi-projective variety W is also valid for any set $U \subseteq W$ that is open in W.
2. Let $\{U_i\}$ be the standard cover of $\mathbf{P}^n$ by copies of $\mathbf{A}^n$ defined in § 1.2. Then for each i, $W \cap U_i$ is a quasi-affine variety. Indeed, write $W = V \cap X$ for X open in $\mathbf{P}^n$. Then we have

$$W \cap U_i = (V \cap U_i) \cap (X \cap U_i). \tag{1}$$

   By § 2.3, $V \cap U_i$ is an affine variety; and X and U_i are open in $\mathbf{P}^n$. Therefore by (1), $W \cap U_i$ is an open subset of the affine variety $V \cap U_i$.

6.3. General Varieties

Hereafter, when we use the term **variety** without qualification, we shall mean any one of an affine, quasi-affine, projective, or quasi-projective variety. A **closed** variety will mean an affine or projective variety.

A variety is **irreducible** if its closure is irreducible as defined in § 2.4. From § 3.3, we see that every variety is a finite union of irreducible components.

7. Coordinate Rings; Regular Functions and Local Rings

In this section we define the coordinate ring of an affine or projective variety. We also define the related concepts of regular functions and local rings for varieties.

7.1. Affine and Quasi-Affine Varieties

In this section, $V \subseteq \mathbf{A}^n$ is an affine variety.

A **coordinate function** on V is a function $p|_V : V \to K$ obtained by choosing a polynomial p in $K[z]$, considering p as a function from $\mathbf{A}^n$ to K, and restricting the domain of p to V.

We define the **coordinate ring** of V, written $K[V]$, as follows:

$$K[V] = K[z]/I(V).$$

That is, $K[V]$ is the set of polynomials p in $K[z]$, modulo the relation that $p \sim q$ if $p - q$ is in the vanishing ideal $I(V)$, i.e., $(p-q)(a) = 0$ for all $a \in V$. This is true if and only if $p|_V = q|_V$. Therefore the elements of $K[V]$ are exactly the coordinate functions on V.

In order to simplify the wording, we will refer to elements of $K[V]$ as **polynomials** even though they are really equivalence classes of polynomials. For example, when we say "the polynomial p in $K[V]$" we will mean "the equivalence class $[p] = p + I(V)$ in $K[V]$ of the polynomial p in $K[z]$."

The maximal ideals of $K[V]$ correspond exactly to the vanishing ideals of points of V. That is, an ideal $I \subseteq K[V]$ is maximal if and only if, for some $a \in V$, I is the ideal $I(\{a\})/I(V)$ of coordinate functions in $K[V]$ that vanish at a (proof omitted). Note the following:

1. I is the ideal generated by the polynomials $\{z_i - a_i\}$, where $\{a_i\}$ are the coordinates of a. For example, in $\mathbf{A}^2$, the ideal $I(\{a\})$ for $a = (1,2)$ is generated by the polynomials $\{z_1 - 1, z_1 - 2\}$.
2. In $\mathbf{A}^1$, this statement follows from the fundamental theorem of algebra: for K algebraically closed, every monic polynomial in $K[z_1]$ is a product of factors $\{z_1 - k_j\}$. Since $K[z_1]$ is a principal ideal domain, the maximal ideals are exactly the principal ideals $(z_1 - k)$ for $k \in K$.

We write $\mathbf{m}_a$ to denote the maximal ideal of $K[V]$ corresponding to the point a.

Let $U \subseteq V$ be a quasi-affine variety (§ 6.1). We define the **coordinate ring** $K[U]$ to be the coordinate ring $K[W]$, where W is the closure of U.

Let $U \subseteq V$ be a quasi-affine variety, and let $f : U \to K$ be a function.

1. For any point a in U, we say that f is **regular at** a if there exist a set $W \subseteq U$ open in U containing a and polynomials p and q in $K[z]$ such that, for all b in W, (a) $q(b) \neq 0$ and (b) $f(b) = p(b)/q(b)$. In this case we say that f is **represented by** the expression p/q on W.
2. We say that f is **regular on** U if it is regular at every point a in U.
3. Let p and q be polynomials in $K[z]$. Then there exists a set $W \subseteq U$, open in U, such that $[p]/[q]$ represents a regular function $f : W \to K$. Indeed, let X be the set of points a in U such that $q(a) = 0$. Then X is an affine variety, so its complement $\mathbf{A}^n - X$ is open. Let $W = U \cap (\mathbf{A}^n - X)$. Then W is open, and $q(a) \neq 0$ for a in W, so $[p]/[q]$ is regular on W.

By taking $U = V$, we can apply the definition of a regular function to the affine variety V. In this case the function $f : V \to K$ is regular on V if and only if f is represented by $p/1$ on V for some polynomial p in $K[V]$ (i.e., is represented by $(p + qI(V))/1$ for some polynomial p and all polynomials q in $K[z]$) (proof omitted).

Let U be a distinguished open subset $V - V(q)$ for some polynomial q in $K[z]$ (see § 5.1).

1. Let R be the set of functions regular on U. It is a ring under the addition rule
$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}.$$
2. Let $[q]$ be the equivalence class of q in $K[V]$. The set $S = \{1, [q], [q]^2, \dots\}$ of powers of $[q]$ is a multiplicative monoid. Therefore we may form the ring of fractions $S^{-1}K[V]$. See *Definitions for Commutative Algebra*, § 14. This ring is exactly the ring R (proof omitted).

Fix a point $a \in V$.

1. As noted above, the ideal of polynomials in $K[V]$ vanishing at a is the maximal ideal $\mathbf{m}_a$.
2. Every maximal ideal is prime, so $\mathbf{m}_a$ is prime, and therefore its complement $S = K[V] - \mathbf{m}_a$ is a multiplicative monoid. See *Definitions for Commutative Algebra*, § 14. S is the set of polynomials that do not vanish at a.

3. We may form $S^{-1}K[V]$, which is the localization of $K[V]$ with respect to the maximal ideal $\mathbf{m}_a$. See *Definitions for Commutative Algebra*, § 14. This ring is called the **local ring of** V **at** a and denoted $\mathbf{O}_a(V)$. It contains all expressions p/q with p and q in $K[V]$ and $q(a) \neq 0$, where p'/q' represents the same element if $pq' - p'q$ is a zero divisor in $K[V]$. If V is irreducible, then $K[V]$ is an integral domain; in this case $pq' - p'q$ is a zero divisor if and only if $pq' - p'q = 0$, i.e., $pq' = p'q$.
4. Let F be the set of all pairs (U, f), where U is open in V, U contains a, and $f: U \to K$ is a regular function. Construct a set G from F by identifying pairs of elements that satisfy the following equivalence relation: $(U, f) \sim (W, g)$ if there exists a set $X \subseteq U \cap W$ such that $a \in X$, and f and g agree on X. The elements of G are called **regular function germs**.[1]
5. If V is irreducible, then the set G is exactly the ring $\mathbf{O}_a(V)$. Indeed, a regular function germ at a is represented by an expression p/q, with p and q in $K[V]$ and $q(a) \neq 0$. Another expression p'/q' represents the same germ if

$$p'(b)/q'(b) = p(b)/q(b)$$

   for all b in an open neighborhood of a. This is true if and only if $pq' - p'q = 0$ in $K[V]$. But this is exactly the rule for constructing the ring $S^{-1}K[V]$.

7.2. Projective and Quasi-Projective Varieties

In this section, $V \subseteq \mathbf{P}^n$ is a projective variety.

We define the **coordinate ring** of V, written $K[V]$, as follows:

$$K[V] = K[Z]/I(V).$$

That is, $K[V]$ is the set of polynomials p in $K[Z]$, modulo the relation that $p \sim q$ if $p - q$ is in the homogeneous ideal $I(V)$. Again we call the elements of $K[V]$ **polynomials** and freely write p instead of $[p]$ to refer to elements of $K[V]$ represented by polynomials p in $K[Z]$. Note that, unlike in the affine case discussed in § 7.1, the elements of the coordinate ring $K[V]$ are not well-defined functions on V: in general, $p(a) \neq p(ka)$ where a is a point in $\mathbf{A}^{n+1}$ that represents a point on V.

The polynomial ring $K[Z]$ is a graded ring. See *Definitions for Commutative Algebra*, § 25. It is also a graded $K[Z]$-module, and this graded $K[Z]$-module is a graded K-algebra. The coordinate ring $K[V]$ is also a graded ring, a graded $K[V]$-module, and a graded K-algebra.

Let $U \subseteq V$ be a quasi-projective variety (§ 6.2). We define the **coordinate ring** $K[U]$ to be the coordinate ring $K[W]$, where W is the closure of U.

Let $U \subseteq V$ be a quasi-projective variety, let $f: U \to K$ be a function, and let $\{U_i\}$ be the standard open cover of $\mathbf{P}^n$. We say that f is **regular on** U if, for every i, $f|_{U \cap U_i}$ is regular on $U \cap U_i$ as defined in § 7.1. By 6.2, $U \cap U_i$ is a quasi-affine variety, so the condition of the definition in § 7.1 is satisfied. Note also the following:

1. If the quasi-projective variety U is an affine variety (see § 6.2), then we have $U \subseteq V \cap U_i$, and so $U \cap U_i = U$. In this case the definition stated here coincides with the definition stated in § 7.1.
2. By taking $U = V$, we can apply the definition stated here to projective varieties.
3. Let P and Q be homogeneous polynomials in $K[Z]$ of the same degree d. By the same argument we made in § 7.1, there exists a set $W \subseteq U$, open in U, such that $Q(a) \neq 0$ for a in W. Then for each U_i, $[P]/[Q]$ is well-defined on $W \cap U_i$, because for each a in W we have

$$([P]/[Q])(ka) = P(ka)/Q(ka) = (k^d P(a))/(k^d Q(a)) = P(a)/Q(a).$$

   Therefore $[P]/[Q]$ represents a regular function on W.

[1] If you have not seen the term "function germ" before, you may be confused as to what a function germ is and/or why we need to define such a thing. The idea is to focus on the behavior of a function near a point a, and to abstract away details such as the domain of definition of the function or the behavior of the function on points away from a. An example of a function germ from complex analysis is a power series $P(z)$ that converges in a neighborhood of zero. If we specify a domain of definition U inside the radius of convergence, then we get a function $P(z)|_U: U \to \mathbf{C}$; if we don't specify a domain, then we get a function germ at zero. All the functions $P(z)|_U$ that we could get by specifying a domain U that contains a are in some sense equivalent.

Let U be a distinguished open subset $V - V(Q)$ for some polynomial Q in $K[Z]$.

1. Let R be the set of functions regular on U. As in § 7.1, R is a ring.
2. Let $[Q]$ be the equivalence class of Q in $K[V]$. Form the set $S = \{1, [Q], [Q]^2, \dots\}$ of powers of $[Q]$. Let F be the ring of fractions $S^{-1}K[V]$.
3. Let $R' \subseteq F$ be the subring of elements represented by P/Q', where P and Q' are homogeneous polynomials of the same degree. Then each element of R' defines a regular function on U. Further, $R' = R$ (proof omitted).

Let $W \subseteq V$ be a quasi-projective variety, and fix a point $a \in W$.

1. The **local ring at** a is defined analogously to the definition stated in § 7.1 for the affine case: it is the ring of germs of functions $f: U \to K$, where U is a neighborhood of a that is open in W, and f is regular on U. We denote the local ring at a as $\mathbf{O}_a(W)$.
2. For any i, let X be the closure in $\mathbf{A}^n$ of $U_i \cap W$. Then $\mathbf{O}_a(W) = \mathbf{O}_a(X)$, where $\mathbf{O}_a(X)$ is the local ring of X at a defined in § 7.1 (proof omitted).

Again, by taking $U = V$, we can apply the definition to projective varieties.

8. Regular Maps Between Varieties

In this section we define the concept of a regular map or morphism between varieties.

8.1. Regular Maps to Quasi-Affine Varieties

Let $\phi: S \to K^n$ be a map from a set S to the n-dimensional vector space K^n. The ith **coordinate map** $\phi_i: S \to K$ is the composition of ϕ with the projection π_i onto the ith coordinate of K^n. That is, $\phi_i = \pi_i \circ \phi$. We may represent ϕ as the tuple of maps $(\phi_1, \dots, \phi_n)$, one for each coordinate of K^n.

Let V be a variety.

1. A **regular map** or **morphism** ϕ from V to $\mathbf{A}^n$, written $\phi: V \to \mathbf{A}^n$, is a map from V to $\mathbf{A}^n$, each of whose coordinate maps ϕ_i is a regular function on V (§ 7). For example, let $V \subseteq \mathbf{A}^2$ be the affine variety given by the zero set of $p(z) = z_1^2 - z_2^2$. Then $I(V)$ is the principal ideal $(p) \subseteq K[z]$. Let $\phi_1(z)$ be the equivalence class mod $I(V)$ of the polynomial z_1^2, i.e., the set $\{z_1^2 + pq\}_{q \in K[z]}$. Let $\phi_2(z)$ be the equivalence class mod $I(V)$ of the polynomial z_2^2. Then $\phi(z) = (\phi_1(z), \phi_2(z))$ is a regular map $\phi: V \to \mathbf{A}^2$.
2. Let $W \subseteq \mathbf{A}^n$ be a quasi-affine variety. A regular map $\phi: V \to W$ is a regular map $\phi: V \to \mathbf{A}^n$ whose image is contained in W. If W is an affine variety, each such map corresponds to a ring homomorphism from $K[W]$ to the ring of regular functions on V (proof omitted).

Let V be a variety, let $W \subseteq \mathbf{A}^n$ be a quasi-affine variety, and let $\phi: V \to W$ be a regular map. Then ϕ is continuous, i.e., for every $U \subseteq W$ that is open in W, $\phi^{-1}(W)$ is open in V. To see this, let $p(z_1, \dots, z_n)$ be a polynomial, and let $X_p = W - V(p)$ be the distinguished open set of points in W that are not in the zero set of p. Since every open set in W is a union of distinguished open sets, it suffices to prove that $\phi^{-1}(X_p)$ is open.

1. If V is quasi-affine, then V is covered by open sets Y_j such that on each Y_j, each of the coordinate maps ϕ_i is represented by an expression p_{ij}/q_{ij}. For each j we may construct a polynomial p_j by (a) substituting p_{ij}/q_{ij} for z_i in p and (b) multiplying by sufficiently high powers of all the q_{ij} to clear the denominators. For any j, let a be a point in Y_j. Since $q_{ij}(a) \neq 0$ for all i, $\phi(a)$ is outside the zero set of p if and only if a is outside the zero set of p_j. Therefore for each j, $Y_j \cap \phi^{-1}(X_p)$ is open in V. Since the Y_j cover $\phi^{-1}(X_p)$, we have that $\phi^{-1}(X_p)$ is open in V.
2. If V is quasi-projective, then we may apply the result from item 1 to $V \cap U_i$ for each of the standard open sets U_i.

Let V and W be quasi-affine varieties.

1. A **regular isomorphism** or **isomorphism** between V and W is a pair of regular maps $\phi: V \to W$ and $\psi: W \to V$ such that each map is a bijection, and each is the inverse of the other.
2. We say that V and W are **isomorphic** if there is an isomorphism between them.

The coordinate ring $K[V]$ of an affine variety is an invariant of isomorphism: two affine varieties V and W are isomorphic if and only if their coordinate rings $K[V]$ and $K[W]$ are isomorphic as K-algebras (proof omitted).

Let $\phi\colon \mathbf{A}^n \to \mathbf{A}^n$ be a regular isomorphism each of whose coordinate maps is a polynomial of degree one. We say that ϕ is an **affine coordinate transform** or **affine change of coordinates**.

8.2. Regular Maps to Quasi-Projective Varieties

Let V be a variety.

1. A **regular map** or **morphism** ϕ from V to $\mathbf{P}^n$, written $\phi\colon V \to \mathbf{P}^n$, is a map from V to $\mathbf{P}^n$ such that (a) ϕ is continuous; and (b) for each of the standard open sets $U_i \subseteq \mathbf{P}^n$, the restriction of ϕ to $\phi^{-1}(U_i)$ is a regular map in the sense of § 8.1. To establish (a), since the open sets U_i cover $\mathbf{P}^n$, it suffices to show that $\phi^{-1}(U_i)$ is open for each i.
2. Let $W \subseteq \mathbf{P}^n$ be a quasi-projective variety. A regular map $\phi\colon V \to W$ is a regular map $\phi\colon V \to \mathbf{A}^n$ whose image is contained in W.

Fix a variety V, and let $\{\phi_i\colon V \to K\}_{i\,\in\,[0,n]}$ be a family of regular functions on V.

1. Let $\phi\colon V \to \mathbf{P}^n$ be a regular map. If $\phi(a) = [\phi_0(a), \dots, \phi_n(a)]$ for all a in V, then we say that the regular map ϕ has **regular coordinate functions** ϕ_i.
2. Let $W \subseteq V$ be the set of points a such that $\phi_i(a) \neq 0$ for some i. Then W is open in V, and the map $\phi\colon W \to \mathbf{P}^n$ given by $a \mapsto [\phi_i(a)]$ is a regular map with regular coordinate functions.

Fix variety V, and let $\phi\colon V \to \mathbf{P}^n$ be a regular map. It is not necessarily the case that ϕ has regular coordinate functions on all of V. For example, let $V \subseteq \mathbf{P}^3$ be the zero set of $Z_0^2 + Z_1^2 - Z_2^2$, and let $\phi\colon V \to \mathbf{P}^2$ be the map given by

$$\phi(Z) = \phi_0(Z) = \left[1, \frac{Z_0}{Z_2 + Z_1}\right] \text{ if } Z_2 + Z_1 \neq 0$$

$$\phi(Z) = \phi_1(Z) = \left[\frac{Z_0}{Z_2 - Z_1}, 1\right] \text{ if } Z_2 - Z_1 \neq 0.$$

Note that in this definition, ϕ_0 and ϕ_1 are not coordinate functions; they are alternate definitions on overlapping domains. Then

1. ϕ is a well-defined map. Indeed, in the intersection W of the domains of ϕ_0 and ϕ_1, $Z_0/(Z_2 - Z_1)$ is well-defined. Therefore for Z in W we have
$$\phi_0(Z) = \left[1, \frac{Z_0}{Z_2 + Z_1}\right] \cdot \frac{Z_0}{Z_2 - Z_1} = \left[\frac{Z_0}{Z_2 - Z_1}, \frac{Z_0^2}{Z_2^2 - Z_1^2}\right] = \left[\frac{Z_0}{Z_2 - Z_1}, 1\right] = \phi_1(Z).$$
2. ϕ is defined on all of V. Indeed, for any point a on V, if $a_2 + a_1 = 0$ and $a_2 - a_1 = 0$, then $2a_2 = 0$, so $a_2 = 0$; then $a_1 = 0$ by the domain restriction; then $a_0 = 0$ on V. Therefore $a = [0, 0, 0]$, which is not a point of $\mathbf{P}^3$.
3. Each ϕ_i is the restriction of ϕ to the inverse image $\phi^{-1}(U_i)$, each ϕ_i is continuous because it has regular coordinate functions, and each ϕ_i is a regular function. Therefore ϕ is a regular map. However, each ϕ_i is undefined on $V - \phi^{-1}(U_i)$. For example, $\phi^{-1}(U_0) = \phi^{-1}([0, 1]) = \{[0, 1, -1]\}$, and ϕ_0 is undefined at $[0, 1, -1]$. Further, there is no family of coordinate functions ϕ_i such that the ϕ_i are regular on all of V.

Let V and W be quasi-projective varieties.

1. An **isomorphism** between V and W is a pair of regular maps $\phi\colon V \to W$ and $\psi\colon W \to V$ such that each map is a bijection, and each is the inverse of the other.
2. We say that V and W are **isomorphic** if there is an isomorphism between them.
3. If the coordinate rings $K[V]$ and $K[W]$ are isomorphic as graded K-algebras (see § 7.2), then we say that V and W are **projectively equivalent**. In this case, V and W are isomorphic. However, unlike the affine case (§ 8.1), V and W may be isomorphic without being projectively equivalent (proof omitted).

Let $\phi\colon \mathbf{P}^n \to \mathbf{P}^n$ be a projective equivalence with regular coordinate functions each of which is a homogeneous polynomial of degree one. We say that ϕ is a **projective coordinate transform** or **projective change of coordinates**.

8.3. Intrinsic and Extrinsic Properties of Varieties

Fix a variety V. Let $S(V)$ be a true statement about V. If, for any variety W that is isomorphic to V, $S(W)$ is also true, then we say that $S(V)$ is an **intrinsic property** of V. Otherwise we say that $S(V)$ is an **extrinsic property** of V. An intrinsic property is a property of the equivalence class C of all varieties isomorphic to V. We call C the **isomorphism class** of V or the **abstract variety** corresponding to V. An extrinsic property depends on the embedding of V in affine or projective space.

9. Products of Varieties

In this section we define products of varieties.

9.1. In Affine Space

For any m and n, we may form the Cartesian or set-theoretic product $\mathbf{A}^m \times \mathbf{A}^n$. As a set, this product is equal to $K^m \times K^n$, which naturally has the structure of $K^m \oplus K^n = K^{m+n} = \mathbf{A}^{m+n}$. Therefore we define

$$\mathbf{A}^m \times \mathbf{A}^n = \mathbf{A}^{m+n}.$$

This definition lets us take products of arbitrary affine and quasi-affine varieties. For example, if $V \subseteq \mathbf{A}^m$ and $W \subseteq \mathbf{A}^n$ are affine varieties, then $V \times W$ is the set of all points a in $\mathbf{A}^{m+n}$ such that the first m coordinates define a point of V in $\mathbf{A}^m$ and the last n coordinates define a point of W in $\mathbf{A}^n$. Equivalently, if we write the coordinates of $\mathbf{A}^m$ as $x = (x_1, \dots, x_m)$ and we write the coordinates of $\mathbf{A}^n$ as $y = (y_1, \dots, y_n)$, and if V is generated by the polynomials $F \subseteq K[x]$ and W is generated by the polynomials $G \subseteq K[y]$, then $V \times W$ is the affine variety in $\mathbf{A}^m \times \mathbf{A}^n$ generated by $F \cup G \subseteq K[x, y]$. We handle quasi-affine varieties similarly.

9.2. In Projective Space

The set $S = \mathbf{P}^m \times \mathbf{P}^n$ does not naturally have the structure of a projective variety. Indeed, write a general point of $\mathbf{P}^m$ as $X = [X_0, \dots, X_m]$, and write a general point of $\mathbf{P}^n$ as $Y = [Y_0, \dots, Y_n]$. Then a point in S is a tuple of coordinates $[X], [Y]$ such that $[X], [Y] \sim [k_1 X], [k_2 Y]$ for any $k_1, k_2 \neq 0$ in K; whereas a point in projective space would be a tuple of coordinates $[X, Y]$ such that $[X, Y] \sim [kX, kY]$ for any $k \neq 0$.

To give S the structure of a projective variety, we let $N = mn + m + n$, and we embed S in $\mathbf{P}^N$ as follows:

1. Let the homogeneous coordinates of $\mathbf{P}^N$ be $Z_0, \dots, Z_N$. Note that there are

   $$N + 1 = mn + m + n + 1 = (m + 1)(n + 1)$$

   coordinates. So there is one coordinate for each pair (X_i, Y_j) of coordinates in X and Y.
2. Let $\sigma\colon [0, m] \times [0, n] \to [0, N]$ be given by $\sigma(i, j) = (n + 1)i + j$. σ maps each pair of coordinate indices (i, j) in $\mathbf{P}^m \times \mathbf{P}^n$ uniquely to a coordinate index in $\mathbf{P}^N$.
3. Let $\phi\colon S \to \mathbf{P}^N$ be the regular map with regular coordinate functions (§ 8.2) whose coordinate function $\phi_{\sigma(i,j)}$, for each $i \in [0, m]$ and $j \in [0, n]$, is given by

   $$\phi_{\sigma(i,j)}([X], [Y]) = X_i Y_j. \tag{2}$$

   That is, the homogeneous coordinate at index $\sigma(i, j) = (n + 1)i + j$ in the image of $\phi([X], [Y])$ is $X_i Y_j$.

Note the following:

1. Fix an element a of $\mathbf{P}^m$ and an element b of $\mathbf{P}^n$.
   a. For some i and j, we have $a_i \neq 0$ and $b_j \neq 0$. Therefore $\phi_{\sigma(i,j)} = a_i b_j \neq 0$.
   b. For $k_1, k_2 \neq 0$, we have $\phi([k_1 a], [k_2 b]) = k_1 k_2 \phi([a], [b])$.

   Therefore ϕ is a well-defined map to $\mathbf{P}^N$.
2. ϕ is an injection. To see this, fix a point c in $\phi(S)$, and let a in $\mathbf{P}^m$ and b in $\mathbf{P}^n$ be points such that $\phi(a, b) = c$.
   a. For some i_0 there is a unique representative $[a_i]$ of a such that $a_{i_0} = 1$, and for some j_0 there is a unique representative $[b_j]$ of b such that $b_{j_0} = 1$. The coordinates $\{c_k\}$ with $c_{\sigma(i,j)} = a_i b_j$ provide the unique representative of c with $c_{\sigma(i_0, j_0)} = 1$.

b. Now consider another pair of points a' and b' in $\mathbf{P}^m \times \mathbf{P}^n$, with $a' \neq a$ or $b' \neq b$. We will show that $\phi(a', b') \neq c$.

i. If a' has coordinate zero at i_0 or b' has coordinate zero at j_0, then coordinate $\sigma(i_0, j_0)$ of $\phi(a', b')$ is zero, so $\phi(a', b') \neq c$.

ii. Otherwise, let $[a'_i]$ be the unique representative of a' with $a'_{i_0} = 1$, and let $[b'_j]$ be the unique representative of b' with $b'_{j_0} = 1$. If $a' \neq a$, then $a'_{i_1} \neq a_{i_1}$ for some $i_1 \neq i_0$. In this case

$$\phi_{\sigma(i_1,j_0)}([a'_i],[b'_j]) = a'_{i_1} \neq a_{i_1} = \phi_{\sigma(i_1,j_0)}([a_i],[b_j]).$$

Therefore $\phi(a', b') \neq c$. A similar argument shows that if $b' \neq b$, then $\phi(a', b') \neq c$.

3. $\phi(S) \subseteq \mathbf{P}^N$ is the zero set of the homogeneous polynomials

$$P_{\sigma(i_1,j_1)\sigma(i_2,j_2)}(Z) = Z_{\sigma(i_1,j_1)}Z_{\sigma(i_2,j_2)} - Z_{\sigma(i_1,j_2)}Z_{\sigma(i_2,j_1)} \tag{3}$$

in $K[Z]$, for all pairs of pairs of indices $((i_1, j_1), (i_2, j_2))$ with $i_2 > i_1$ and $j_2 > j_1$.

a. $\phi(S)$ is contained in this zero set: for example, $(a_1 b_2)(a_3 b_4) - (a_1 b_4)(a_3 b_2) = 0$.

b. On the other hand, given any element c in the zero set, we can choose a representative $[c_k]$ with $c_{\sigma(i_0,j_0)} = 1$ for some i_0 and j_0. Let a be the point in $\mathbf{P}^m$ represented by $[a_i]$, where $a_i = c_{\sigma(i,j_0)}$, and let b be the point in $\mathbf{P}^n$ represented by $[b_j]$, where $b_j = c_{\sigma(i_0,j)}$. Then

$$\phi_{\sigma(i,j)}([a_i][b_i]) = c_{\sigma(i,j_0)}c_{\sigma(i_0,j)} = c_{\sigma(i,j)}c_{\sigma(i_0,j_0)} = c_{\sigma(i,j)}.$$

So $\phi(a, b) = c$, and so $\phi(S)$ contains the zero set.

Therefore $\phi(S) \subseteq \mathbf{P}^N$ is a projective variety.

The map ϕ is called the **Segre map** or the **Segre embedding**.

We may express the Segre map in the language of tensor products.

1. Recall that $K^{m+1} \otimes K^{n+1} = K^{(m+1)(n+1)}$, and that a basis for $K^{m+1} \otimes K^{n+1}$ is the set of elements $\{c_i \otimes d_j\}$, where $\{c_i\}$ are basis vectors for K^{m+1} and $\{d_j\}$ are basis vectors for K^{n+1}. See *Definitions for Commutative Algebra*, § 8.
2. We may represent an element $a \in K^{m+1}$ as $\sum_i a_i c_i$, where a_i are the coordinates of a, and c_i are the standard unit basis vectors. Similarly, we may represent an element $b \in K^{n+1}$ as $\sum_j b_j d_j$.
3. The element $a \otimes b \in K^{m+1} \otimes K^{n+1}$ is then

$$(\sum_i a_i c_i) \otimes (\sum_j b_j d_j) = \sum_{i,j} a_i c_i \otimes b_j d_j = \sum_{i,j} (a_i b_j)(c_i \otimes d_j).$$

Identifying the coefficient $a_i b_j$ of $(c_i \otimes d_j)$ with the coordinate $\sigma(i, j)$ of $K^{(m+1)(n+1)}$, we see that this is the image $\phi([a], [b])$ of the Segre map.

4. If we write $\mathbf{P}(\mathbf{A}^{m+1})$ to mean the projective space $\mathbf{P}^m$ with homogeneous coordinates in $\mathbf{A}^{m+1}$, then the Segre map takes the element $([a], [b])$ in $\mathbf{P}(\mathbf{A}^{m+1}) \times \mathbf{P}(\mathbf{A}^{n+1})$ to the element $[a \otimes b]$ in $\mathbf{P}(\mathbf{A}^{(m+1)(n+1)})$.
5. Compare the product of affine spaces (§ 9.1), which takes (a, b) in $\mathbf{A}^m \times \mathbf{A}^n$ to $a \oplus b$ in $\mathbf{A}^{m+n}$.

We may express projective subvarieties of $\phi(S) = \phi(\mathbf{P}^m \times \mathbf{P}^n)$ in terms of polynomials in the ring $K[X, Y]$.

1. Let $P(X, Y)$ be a polynomial in $K[X, Y]$. Let $P_Y(X)$ be the corresponding polynomial in $K[Y][X]$, i.e., with variables in X and coefficients in $K[Y]$, and let $P_X(Y)$ be the corresponding polynomial in $K[X][Y]$. We say that P is **bihomogeneous** if P_X is homogeneous with degree d_X and P_Y is homogeneous with degree d_Y. In this case we say that P has **bidegree** (d_X, d_Y). For example, let $P(X, Y) = X_1 X_2 Y_1^3 + X_1^2 Y_1 Y_2^2$. Then $P_Y(X) = (Y_1^3)X_1X_2 + (Y_1Y_2^2)X_1^2$, $P_X(Y) = (X_1X_2)Y_1^3 + (X_1^2)Y_1Y_2^2$, and P is bihomogeneous with bidegree $(2, 3)$.
2. We say that a bihomogeneous polynomial has **equal bidegree** if its bidegree is (d, d) for some d. For example, $X_1Y_1 + X_2Y_2$ has equal bidegree with $d = 1$.
3. Let V be a projective subvariety of $\phi(S)$. Then V is the intersection of $\phi(S)$ with a subvariety of $\mathbf{P}^N$, i.e., it is in the image (2) and is also in the zero set of a family of homogeneous polynomials $\{P_\alpha(Z)\}$ in $K[Z]$.

Replacing each $Z_{\sigma(i,j)}$ with $X_i X_j$ in each $P_\alpha(Z)$ according to (2) yields a family of polynomials $\{Q_\alpha(X,Y)\}$ in $K[X,Y]$, each of which is bihomogeneous with equal bidegree.

4. On the other hand, given a bihomogeneous polynomial $Q[X,Y]$ of equal bidegree, we can group pairs of factors X_i and Y_j and replace each pair by $Z_{\sigma(i,j)}$; according to the relations (3), it doesn't matter how we do this grouping. The resulting homogeneous polynomial $P(Z)$ is in the image (2).

Items 3 and 4 establish that the subvarieties V of $\phi(S)$ are precisely the images $\phi(W)$ of subsets $W \subseteq \mathbf{P}^m \times \mathbf{P}^n$ cut out by families of homogeneous polynomials in $K[X,Y]$ of equal bidegree.

If $V \subseteq \mathbf{P}^m$ and $W \subseteq \mathbf{P}^n$ are projective varieties, then the Segre image $\phi(V \times W) \subseteq \mathbf{P}^N$ is a projective variety.

1. Suppose that V is generated by homogeneous polynomials $\{P_\alpha(X)\}$, and let V_α denote the zero set of P_α. There is no point in $\mathbf{P}^n$ where all the Y_i are zero. Therefore, for each α, $V_\alpha \times \mathbf{P}^n$ is the zero set of $F_\alpha = \{P_\alpha(X)Y_i^{d_\alpha}\}_{i \in [1,n]}$, where d_α is the degree of P_α.
2. Let $F_V = \bigcup_\alpha F_\alpha$. Then $V \times \mathbf{P}^n$ is the zero set of F_V.
3. Similarly, suppose that W is generated by homogeneous polynomials $\{P_\beta(Y)\}$, and let W_β denote the zero set of P_β. Then for each β, $\mathbf{P}^m \times W_\beta$ is the zero set of $F_\beta = \{P_\beta(Y)X_i^{d_\beta}\}_{i \in [1,m]}$, where d_β is the degree of P_β.
4. Let $F_W = \bigcup_\beta F_\beta$. Then $\mathbf{P}^m \times W$ is the zero set of F_W.
5. $V \times W$ is the zero set of $F = F_V \cup F_W$. All the polynomials in F are bihomogeneous of equal bidegree, and they cut out $V \times W$; so by the previous paragraph, $\phi(V \times W)$ is a subvariety of $\phi(S)$.

A similar result holds for quasi-projective varieties (proof omitted).

We define the **product** of any two quasi-projective varieties $V \subseteq \mathbf{P}^m$ and $W \subseteq \mathbf{P}^n$ to be the Segre image $\phi(V \times W) \subseteq \mathbf{P}^N$. This definition gives us a stronger notion of product than the set-theoretic product, because in this sense the product of two varieties is again a variety.

10. Parameterized Families and General Objects

Let O be a set of objects (not necessarily varieties), and let $p \colon O \to \{\mathsf{true}, \mathsf{false}\}$ be a predicate function that expresses whether a particular property holds for a member o of O. Let V be an irreducible variety, and let $F = \{o_a\}_{a \in V}$ be a family of objects in O.

1. We say that the family F is **parameterized by** the variety V.
2. We say that the **general object** of F has property p if $p(o_v) = \mathsf{true}$ for all v on an open dense subset of V.

For example, consider the set of all lines in $\mathbf{P}^2$. Each line is given by the zero set of a homogeneous polynomial

$$L(Z) = a_0 Z_0 + a_1 Z_1 + a_2 Z_2,$$

with not all the a_i equal to zero. We may identify each such line with a point $a = [a_0, a_1, a_2]$ in $\mathbf{P}_2$. Thus we may express the set of lines in $\mathbf{P}_2$ as a family $\{L_a\}_{a \in \mathbf{P}^2}$. Fix a point b in $\mathbf{P}^2$, say $b = [0,0,1]$. The lines L_a that do not intersect b are given by points a with $a_2 \neq 0$; the set of all such points is open in the irreducible variety $\mathbf{P}^2$ and hence is dense in $\mathbf{P}^2$ (§ 5.2). Therefore we may say that the general line in $\mathbf{P}^2$ does not intersect the point b.

11. Rational Functions and Maps

In this section we define the concepts of rational functions on varieties and rational maps between varieties. These concepts generalize the concepts of regular functions (§ 7) and regular maps (§ 8) on irreducible varieties.

11.1. Rational Functions

Fix an irreducible affine variety $V \subseteq \mathbf{A}^n$.

1. The coordinate ring $K[V]$ of V is an integral domain, so we may form its field of fractions. See *Definitions for Commutative Algebra*, § 14. We denote the field of fractions $K(V)$ and call it the **rational function field** of V. The elements of $K(V)$ are called **rational functions** on V.
2. For any point a in V, the local ring $\mathbf{O}_a(V)$ is a subring of $K(V)$: $K(V)$ contains all elements p/q with p and q in $K[V]$; whereas $\mathbf{O}_a(V)$ contains all elements p/q in $K(V)$ such that $q(a) \neq 0$.

3. Fix a set $U \subseteq V$ that is open in V.
   a. Let $f: U \to K$ be a regular function (§ 7.1). Then for every point a in U, there is a set $W_a \subseteq U$ that contains a and is open in U such that, on W_a, f is represented by a rational function p/q in $K(V)$. This statement follows from the definitions of a regular function and of a rational function.
   b. Fix a rational function p/q. Then p/q is not necessarily regular on U, because at some point a in U we may have $q(a) = 0$. p/q is regular on a subset of U that is open in U (the complement in U of the zero locus of q).

Fix a quasi-affine variety W. We define the **rational function field** $K(W)$ to be $K(V)$, where V is the closure of W.

Note the mismatch of terminology: a regular function on U is a well-defined function $f: U \to K$ that is represented at each point in U by a ratio of polynomials. A rational function on V is a ratio of polynomials that may or may not be a well-defined function at any point of V. This mismatch is unfortunate, but standard.

Fix an irreducible projective variety $V \subseteq \mathbf{P}^n$.

1. We define the **rational function field** of V, again noted $K(V)$, to be the rational function field $K(U)$ of the affine variety $U = V \cap U_i$ for any of the standard open sets U_i; the field so obtained is independent of i (proof omitted).
2. $K(V)$ is equal to the field of elements $P(Z)/Q(Z)$, where P and Q are homogeneous polynomials in $K[V]$ of the same degree (proof omitted).

Fix a quasi-projective variety W. We define the **rational function field** $K(W)$ to be $K(V)$, where V is the closure of W.

11.2. Rational Maps

In this section, V is an irreducible variety, and W is a variety.

Let S be the set of pairs (U, ψ), where $U \subseteq V$ is nonempty and is open in V, and $\psi: U \to W$ is a regular map (§ 8). Let ~ be the following binary relation on S:

$$(U_1, \psi_1) \sim (U_2, \psi_2) \text{ if } \psi_1 \text{ and } \psi_2 \text{ agree on } U_1 \cap U_2.$$

It is easy to see that ~ is an equivalence relation on S. Let M be the set of equivalence classes of S modulo ~. We call the elements ϕ of M **rational maps**. We write $\phi: V \dashrightarrow W$ to denote a rational map from V to W. Let $\phi = \{(U_i, \psi_i)\}$ be an element of M, and note the following:

1. Any regular map $\psi: V \to W$ induces a rational map $\phi: V \dashrightarrow W$ given by $\phi = \{U_i, \psi|_{U_i}\}$, where the sets U_i are the nonempty subsets of V that are open in V.
2. Let $W = \cap U_i$. The closure of W is the intersection of the closures of the U_i. Since the U_i are nonempty and V is irreducible, each U_i is dense in V (§ 5.1), i.e., the closure of each U_i is V. Therefore the closure of W is V, i.e., W is a nonempty and dense open subset of V set on which all of the ψ_i agree.
3. Let $U = \cup U_i$. Then U is open in V. For each point a in U, define $\psi(a) = \psi_i(a)$, for any i such that X_i contains a. Since the regular maps ψ_i agree on their points of common definition, ψ is well-defined on all of U and is a regular map from U to W. The pair (U, ψ) is therefore an element (U_j, ψ_j) of ϕ. Moreover, for every i, $U_i \subseteq U$; and U is the unique set U_j with this property. U is called the **domain of regularity** of the rational map ϕ. $V - U$ is called the **indeterminacy locus** of ϕ.
4. A rational map ϕ is not a map in the ordinary sense. Instead, it is an equivalence class of maps. In this respect a rational map is similar to a function germ (§ 7.1). This is another bit of idiosyncratic terminology.

The following facts motivate the term "rational map."

1. Assume that $V \subseteq \mathbf{A}^m$ and $W \subseteq \mathbf{A}^n$ are quasi-affine varieties. Let $\{\phi_i\}_{i \in [1,n]}$ be a family of rational functions f_i in $K(V)$. We may represent each f_i as a ratio p_i/q_i of polynomials in $K[z]$ having no common factors of degree greater than zero. Assume that at each point a of W, either (a) $q_i(a) = 0$ for some i or (b) the point $\{p_i(a)/q_i(a)\}$ lies in W. Then the set of points for which (a) is true is the intersection with V of the union of the zero sets of the polynomials q_i in $K[z]$. Therefore the set of points for which (b) is true is a set $U \subseteq V$ that is open in V. Let $\phi(a) = \{\phi_i(a)\}$. On U, each ϕ_i is a regular function, so ϕ is a regular map. Therefore ϕ defines a rational map from V to W.

2. Assume that $V \subseteq \mathbf{P}^m$ and $W \subseteq \mathbf{P}^n$ are quasi-projective varieties. Let $\{\phi_i\}_{i \in [0,n]}$ be a family of rational functions F_i in $K(V)$. We may represent each F_i as a ratio P_i/Q_i of homogeneous polynomials in $K[Z]$ of equal degree, having no common factors of degree greater than zero. Assume that at each point a of W, either (a) $Q_i(a) = 0$ for some i or (b) $P_i(a) = 0$ for all i or (c) the point $[P_i(a)/Q_i(a)]$ lies in W. Let $\phi(a) = [P_i(a)/Q_i(a)]$ on the set of points a where (c) is true. Then by a similar argument to the one made in item 1, ϕ defines a regular map on an open set U, and hence a rational map from V to W.
3. We may reason similarly in the cases where V is quasi-affine and W is quasi-projective, or vice versa. In each case, a family of rational functions defines a rational map.

A rational map $\phi\colon V \dashrightarrow W$ is **dominant** if, for some pair (U, ψ) in ϕ, $\psi(U)$ is dense in W. This is true if and only if it is true for every pair (U, ψ) in ϕ (proof omitted).

11.3. Rational Functions as Rational Maps

Let V be an irreducible variety, and let f be a rational function in $K(V)$.

1. If the closure of V is an affine variety, then $f = p/q$, where p and q are polynomials in $K[z]$ having no common factors. p/q defines a regular function ψ on the open set $U = V - V(q)$.
2. If the closure of V is a projective variety, then $f = P/Q$, where P and Q are homogeneous polynomials in $K[Z]$ with equal degree, having no common factors. P/Q defines a regular function ψ on the open set $U = V - V(Q)$.

In either case, (U, ψ) defines a rational map $\phi\colon V \dashrightarrow \mathbf{A}^1$.

Let F be the set of rational maps $\phi\colon V \dashrightarrow \mathbf{A}^1$. F is a field with operations

$$(U_1, \psi_1) + (U_2, \psi_2) = (U_1 \cap U_2, \psi_1 + \psi_2)$$

$$-(U, \psi) = (U, -\psi)$$

$$(U_1, \psi_1) \cdot (U_2, \psi_2) = (U_1 \cap U_2, \psi_1 \cdot \psi_2)$$

$$1/(U, \psi) = (\psi^{-1}(\mathbf{A}^1 - \{0\}), 1/\psi)$$

F is isomorphic to the rational function field $K(V)$ (proof omitted).

11.4. Graphs and Images of Rational Maps

Let $V \subseteq \mathbf{P}^m$ be a quasi-projective variety, and let $\psi\colon V \to \mathbf{P}^n$ be a regular map. The **graph** of ψ, denoted Γ_ψ, is the subset of $V \times \mathbf{P}^n$ consisting of all pairs (a, b) such that $b = \psi(a)$. Let $S\colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N$ be the Segre map (§ 9.2). If V is a projective variety in $\mathbf{P}^m$, then $S(\Gamma_\psi)$ is a projective variety in $\mathbf{P}^N$ (proof omitted).

Let $V \subseteq \mathbf{P}^m$ be an irreducible variety, and let $\phi\colon V \dashrightarrow \mathbf{P}^n$ be a rational map.

1. The **graph** of ϕ, denoted Γ_ϕ, is the closure of the graph Γ_ψ of the regular map $\psi\colon U \to \mathbf{P}^n$, where (U, ψ) is any representative of the class ϕ. This definition is independent of the representative chosen (proof omitted).
2. Let $\pi_1\colon \Gamma_\phi \to V$ be the projection $S(a, b) \mapsto a$. This notation means that $\pi_1(c) = a$, where a and b are the unique values such that $S(a, b) = c$. Let $\pi_2\colon \Gamma_\phi \to \mathbf{P}^n$ be the projection $S(a, b) \mapsto b$.
   a. The **image** of ϕ is the set $\pi_2(\Gamma_\phi)$. If V is a projective variety, then the image of ϕ is also a projective variety (proof omitted).
   b. Let $X \subseteq V$ be a variety. The **image** of X under ϕ, written $\phi(X)$, is $\pi_2(\pi_1^{-1}(X))$. Note that when $X = V$, we get the definition in part (a).
   c. Let $Y \subseteq \mathbf{P}^n$ be a variety. The **inverse image** of Y under ϕ, written $\phi^{-1}(Y)$, is $\pi_1(\pi_2^{-1}(Y))$.

   Note that these "images" and "inverse images" of rational "maps" do not behave like images and inverse images of ordinary maps. For example, it is not true in general that for any point b in the image of ϕ, there exists a point a in V such that $\phi(a) = b$. So we have more idiosyncratic terminology.

Let V be an irreducible variety, let W be a quasi-projective variety, and let $\phi\colon V \dashrightarrow W$ be a rational map. From the definitions, we have the following:

1. The image of ϕ is contained in the closure of W.
2. ϕ is dominant (§ 11.2) if and only if the image of ϕ is the closure of W.

11.5. Composition of Rational Maps

In this section, $\phi_1 \colon V \dashrightarrow W$ and $\phi_2 \colon W \dashrightarrow X$ are rational maps.

Suppose there exist pairs (U_1, ψ_1) representing ϕ_1 and (U_2, ψ_2) representing ϕ_2 such that $\psi_1(U_1) \cap U_2$ is nonempty. Then $U = \psi_1^{-1}(U_2) \subseteq U_1$ is nonempty; and because ψ_1 is continuous (§ 8), U is open in V. Let ψ be the composition of the restriction of ψ_1 to U with ψ_2, i.e., $\psi = \psi_2 \circ \psi_1|_U$. Then ψ defines a regular map from U to X, so (U, ψ) represents a rational map $\phi \colon V \dashrightarrow X$. We call ϕ the **composition** of ϕ_1 and ϕ_2 and write $\phi = \phi_2 \circ \phi_1$.

1. If, for every pair (U_1, ψ_1) and (U_2, ψ_2), $\psi_1(U_1) \cap U_2$ is empty, then $U = \phi_1^{-1}(U_2)$ is empty, and the composition $\phi_2 \circ \phi_1$ is not defined.
2. If ϕ_1 is dominant (§ 11.2), then $U \neq \varnothing$, and $\phi_2 \circ \phi_1$ is defined.

Assume that ϕ_1 is dominant, and let f be an element of $K(W)$. Then $f \circ \phi_1$ is defined and is a rational map from V to $\mathbf{A}^1$, so it is an element of $K(V)$ (§ 11.3). Let $\phi_1^* \colon K(W) \to K(V)$ be the map that takes f to $\phi_1^* f = f \circ \phi_1$. Then ϕ_1^* is an inclusion of function fields (proof omitted).

11.6. Birational Isomorphism

Let V and W irreducible varieties. Suppose there exist rational maps $\phi_1 \colon V \dashrightarrow W$ and $\phi_2 \colon W \dashrightarrow V$ such that the compositions $\phi_2 \circ \phi_1$ and $\phi_1 \circ \phi_2$ are both defined and equal to the identity map on their domains of regularity. In this case we say that each of the maps ϕ_1 and ϕ_2 is **birational**, and we say that the varieties V and W are **birationally isomorphic** or **birational**. Note that when ϕ_1 and ϕ_2 are birational, each ϕ_i is a dominant rational map.

Fix irreducible varieties V and W.

1. As noted in § 11.2, a regular map $\psi \colon V \to W$ induces a rational map $\phi \colon V \dashrightarrow W$. If ψ is a regular isomorphism (§ 8), then ϕ is a birational isomorphism.
2. Let $\phi \colon V \dashrightarrow W$ be a rational map, and assume that the field K has characteristic zero (see *Definitions for Commutative Algebra*, § 6). Then ϕ is birational if and only if for the general point a of W (§ 10), the inverse image $\phi^{-1}(a)$ (§ 11.4) consists of a single point (proof omitted).
3. V and W are birational if (a) $K(V)$ is isomorphic to $K(W)$ or (b) there exist nonempty open subsets $X \subseteq V$ and $Y \subseteq W$ such that X and Y are isomorphic (§ 8). Conditions (a) and (b) are equivalent (proof omitted).

Let V be a variety. We say that V is **rational** if any of the following conditions holds:

1. V and $\mathbf{P}^n$ are birational. Note this condition implies that V is irreducible.
2. $K(V)$ is isomorphic to the field of fractions of $K[z]$.
3. V contains an open subset U that is isomorphic to an open subset of $\mathbf{A}^n$.

These conditions are equivalent (proof omitted). If these conditions do not hold, then we say that V is **irrational**.

11.7. Rational Maps of Finite Degree

Let V be an irreducible variety, and let W be a variety. Let $\phi \colon V \dashrightarrow W$ be a dominant rational map, and let $\phi^* \colon K(W) \to K(V)$ be the corresponding inclusion of function fields (§ 11.5).

1. $\phi^{-1}(a)$ is finite for the general point of W if and only if $K(V)$ is a finite extension of $\phi^* K(W)$ (see *Definitions for Commutative Algebra*, § 19) (proof omitted).
2. If the condition in item 1 is satisfied, then
    a. We say that ϕ is **generically finite** or of **finite degree**.
    b. The number of points in $\phi^{-1}(a)$ for general a is called the **degree** of the map.
    c. If K has characteristic zero, then the degree of the map is equal to the degree of the field extension (proof omitted).

12. Blowing Up Varieties Along Subvarieties

In this section we define a construction called the blowup of a variety along a subvariety.

12.1. In Affine Space

Fix an irreducible affine variety $V \subseteq \mathbf{A}^m$ and a variety $W \subseteq V$. Let $\{p_i(z)\}_{i \in [1,n]}$ be a family of polynomials in $K[z]$ that generate $I(V)$. Let $\phi\colon V \dashrightarrow \mathbf{P}^n$ be the rational map represented by (X, ψ), where $X = V - W$, and

$$\psi(z) = [p_i(z)].$$

Then we have the following:

1. For every point a in X, at least one of the values $p_i(a)$ is not zero. Therefore $\psi\colon X \to \mathbf{P}^n$ is a regular map with regular coordinate functions (§ 8.2).
2. For every point in W, all the values $p_i(a)$ are zero. Therefore the domain of regularity of ϕ is X, and the indeterminacy locus is W.

Let $Y \subseteq \mathbf{P}^m \times \mathbf{P}^n$ be the set of points $([1,a], b)$ such that $a \in X$ and $b = \psi(a)$, and let $S\colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N$ be the Segre map (§ 9.2). Observe the following:

1. The map from X to Y given by $a \mapsto ([1,a], \psi(a))$ is a bijection. Indeed, for any two points $b = ([1,a], \psi(a))$ and $b' = ([1,a'], \psi(a'))$ in Y, if $a = a'$, then $\psi(a) = \psi(a')$, and $b' = b$. Therefore if $b' \neq b$, then $a' \neq a$.
2. The map in item 1 induces a map $\eta\colon X \to S(Y)$ given by $\eta(a) = S([1,a], \psi(a))$. η is a regular isomorphism (proof omitted).
3. $S(Y)$ is a subset of the graph Γ_ϕ (§ 11.4). Let $\pi\colon \Gamma_\phi \to V$ be the projection map $S([1,a], b) \mapsto a$. Then π restricted to $S(Y)$ is the isomorphism $\eta^{-1}\colon Y \to X$.

The pair (Γ_ϕ, π) is called the **blowup** of V along W. Let E be the set $\pi^{-1}(W)$ of elements $S(a,b)$ in Γ_ϕ such that $a \in W$. E is called the **exceptional divisor** associated with the blowup.

For example, let $V = \mathbf{A}^2$, and let W be the variety $\{(0,0)\}$ generated by $\{z_1 = 0, z_2 = 0\}$. Let $\phi\colon \mathbf{A}^2 \dashrightarrow \mathbf{P}^1$ be the rational map represented by (X, ψ), where $X = \mathbf{A}^2 - W$, and

$$\psi(z_1, z_2) = [z_1, z_2].$$

Then we have the following:

1. X is the set of points $a = (a_1, a_2)$ such that $a \neq (0,0)$.
2. $S(Y) \subseteq S(\mathbf{P}^2 \times \mathbf{P}^1)$ is the set of points $S([1,a],[a]) = S([1,a_1,a_2],[a_1,a_2])$ such that $a \neq (0,0)$.
3. X is isomorphic to $S(Y)$ via the isomorphism $\eta(a) = S([1,a],[a])$.
4. The graph Γ_ϕ is the closure in $\mathbf{P}^N$ of $S(Y)$. Because the open set $U_0 = \{[1,a]\}_{a \in \mathbf{A}^2}$ is dense in $\mathbf{P}^2$, Γ_ϕ is the set $S(Y) \cup E$, where E is the set of points $S([0,a],[a])$ for $a \neq (0,0)$.
5. E is the exceptional divisor $\pi^{-1}(W)$. It is isomorphic to $\mathbf{P}^1$.

Notice how the blowup extends the regular map ψ in a natural way. When $a \neq 0$, the projection of $\pi^{-1}(a)$ onto $\mathbf{P}^1$ agrees with $\psi(a)$ and yields the point $[a]$ of $\mathbf{P}^1$ corresponding to a. When $a = 0$, ψ is undefined, and the projection of $\pi^{-1}(a)$ onto $\mathbf{P}^1$ yields all of $\mathbf{P}^1$.

12.2. In Projective Space

Let I be an ideal of $K[Z]$. We define the **saturation** of I, denoted $\bar{I}$, to be the set of all polynomials p in $K[Z]$ such that, for some $k \geq 0$ and all homogeneous polynomials P of degree $d \geq k$, Pp lies in I.

Fix ideals I and J of $K[Z]$. The following conditions on I and J are equivalent (proof omitted):

1. $\bar{I} = \bar{J}$.
2. Denote by I_m the set of elements of I, each of whose terms has degree at least m. Then for some $M \geq 0$ and all $m \geq M$, $I_m = J_m$.
3. For each i in $[0,n]$, let I_i be the result of converting each polynomial $P(Z_0, \ldots Z_n)$ in I to a polynomial $p(z_1, \ldots, z_n)$ in $K[z]$ as described in § 2.3. Let J_i be the result of converting each polynomial of J in the same way. Then $I_i = J_i$. Therefore, for each i, the zero sets of I and J in $\mathbf{P}^n$ have the same intersection with

the standard open set U_i.

Fix an irreducible projective variety $V \subseteq \mathbf{P}^m$ and a variety $W \subseteq V$. Let $F = \{P_i(z)\}_{i \in [1,n]}$ be a family of homogeneous polynomials in $K[Z]$ of the same degree. Let I be the ideal generated by F, and assume that $\bar{I} = I(V)$. Let $\phi\colon V \dashrightarrow \mathbf{P}^n$ be the rational map represented by (X, ψ), where $X = V - W$, and

$$\psi(z) = [P_i(z)].$$

As in the affine case (§ 12.1), $\psi\colon X \to \mathbf{P}^n$ is a regular map with regular coordinate functions, and the domain of regularity of ϕ is X. Let $\pi\colon \Gamma_\phi \to V$ be the projection map. Again the pair (Γ_ϕ, π) is called the **blowup** of V along W, and $E = \pi^{-1}(W)$ is called the **exceptional divisor** associated with the blowup.

Let $Y \subseteq \mathbf{P}^m \times \mathbf{P}^n$ be the image of X under the map $a \mapsto (a, \psi(a))$. Let $S\colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N$ be the Segre map. Then the map $\eta\colon X \to S(Y)$ given by $\eta(a) = S(a, \psi(a))$ is a regular isomorphism (proof omitted).

12.3. Factoring Rational Maps

Let V be an irreducible closed variety, and let $\phi\colon V_1 \dashrightarrow \mathbf{P}^n$ be a rational map. Then there exist a sequence of varieties $\{V_i\}_{i \in [1,m]}$, subvarieties $W_i \subseteq V_i$, maps $\pi_i\colon V_{i+1} \to V_i$, and a regular map $\psi\colon V_m \dashrightarrow \mathbf{P}^n$ such that the following hold:

1. For each i in $[1, m-1]$ and each $X_i = V_i - W_i$,
    a. (V_{i+1}, π_i) is the blowup of V_i along W_i.
    b. There exists an injective regular map $\eta_i\colon X_i \to V_{i+1}$ such that π_i restricted to $\eta_i(X_i)$ is η_i^{-1}.
2. The composition of regular maps $\psi \circ \eta_{m-1} \circ \cdots \circ \eta_1$ is a regular map $\chi\colon X_1 \to \mathbf{P}^n$. The rational map from V to $\mathbf{P}^n$ represented by (X_1, χ) is ϕ.

(Proof omitted.)

See Figure 1. In this figure, the arrows $X_i \to V_i$ are the inclusion maps. All the solid arrows commute, and (X_1, χ) represents ϕ.

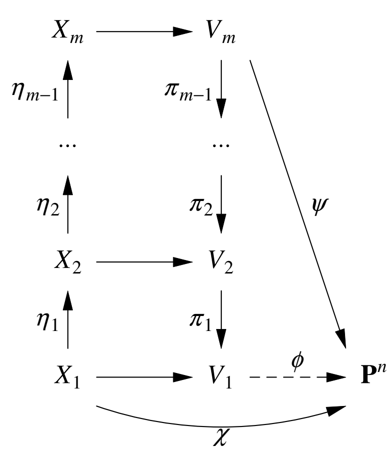

Figure 1: Factoring a rational map as a sequence of blowups.

In the example given in § 12.1, we may take $m = 2$ and $n = 1$. Then

- $V_1 = \mathbf{A}^2$, and $X_1 = \mathbf{A}^2 - \{(0,0)\}$.
- X_2 is the set of points $S([1,a],[a])$ such that $a \neq (0,0)$. Therefore X_2 is isomorphic to $\mathbf{P}^1$.
- V_2 is X_2 together with the points $S([0,a],[a])$ such that $a \neq (0,0)$. Therefore V_2 is isomorphic to $\mathbf{P}^2 - \{[1,0,0]\}$.
- $\psi\colon V_2 \to \mathbf{P}^1$ is the regular map given by $S([z_0, z_1, z_2], [z_1, z_2]) \mapsto [z_1, z_2]$. Notice that ψ is well-defined on V_2, because we never have $z_1 = z_2 = 0$.

- For any point a in $\mathbf{A}^2 - \{(0,0)\}$, applying η_1 yields $S([1,a],[a])$, and then applying ψ yields $[a]$. So $\psi \circ \eta_1$ agrees with ϕ on X_1.

13. The Dimension of a Variety

We now come back to the dimension of a variety, for which we gave an initial definition in § 4.

Let V be an irreducible variety. For any ring A, let dim A denote the Krull dimension of A (see *Definitions for Commutative Algebra*, § 20). Here we take the dimension of the zero ring to be -1. The following definitions of dim V are mutually equivalent (proof omitted):

1. dim V is the dimension of the closure of V according to the definition given in § 4 for a closed variety.
2. dim V is the smallest integer $d \geq -1$ such that the general $(n-(d+1))$-plane (§ 4.4) is disjoint from V. Here we take the empty variety to be a -1-plane. For example:
   a. The dimension of the empty variety $\varnothing$ is -1 in $\mathbf{P}^n$, because the general m-plane is disjoint from $\varnothing$ for $m = n - (-1+1) = n$.
   b. A point a has dimension zero in $\mathbf{A}^3$, because for $m = 3-(0+1) = 2$, the general m-plane is disjoint from a, but for $m = 3-(-1+1) = 3$ the general m-plane (i.e., $\mathbf{A}^3$) is not disjoint from a.
   c. $\mathbf{A}^3$ has dimension 3, because for $m = 3-(3+1) = -1$, the general m-plane (i.e., the empty variety) is disjoint from $\mathbf{A}^3$, but for $m = 3-(2+1) = 0$, the general m-plane (i.e., the general point) is not disjoint from $\mathbf{A}^3$.
3. Let $K[V]$ denote the coordinate ring of V (§ 7). Then dim V = dim $K[V]$.
4. Let a be any point of V, and let $\mathbf{O}_a(V)$ denote the local ring of V at a (§ 7). Then dim V = dim $\mathbf{O}_a(V)$. This definition is independent of the choice of the point a (proof omitted).
5. Let $K(V)$ denote the field of fractions of $K[V]$ (§ 11). Let d be -1 if $K(V) = 0$; otherwise let d be the transcendence degree of $K(V)$ over K (see *Definitions for Commutative Algebra*, § 19). Then dim $V = d$.

Let V be a nonempty variety, and let $\{V_i\}$ be its irreducible components (§ 6.3). As in § 4.3, we define the following:

1. The dimension of V is the maximum of the dimensions of the V_i. If all the V_i have the same dimension d, then we say that V has **pure dimension** d.
2. Let a be a point of V. We define the **local dimension** of V at a, written $\dim_a V$, to be the maximum of the dimensions of the irreducible components V_i that contain a.

In the rest of this section, V is a variety of dimension $d \geq 0$. We assume without loss of generality that $V \subseteq \mathbf{P}^n$; if V is affine or quasi-affine, we have $V \subseteq U_i \subseteq \mathbf{P}^n$, where U_i is one of the standard copies of $\mathbf{A}^n$ in $\mathbf{P}^n$ (§ 1.2).

The **codimension** of V, denoted codim V, is $n-d$.

Let $W \subseteq \mathbf{P}^n$ be a hypersurface (§ 4.4), and let $X = V \cap W$.

1. If W contains an irreducible component of V, then dim $X = d$.
2. Otherwise dim $X = d-1$.

(proof omitted).

Let $\{P_i\}_{i \in [1,k]}$ be a minimal set of homogeneous generators for the closure of V. If $k = n-d$, then we say that V is a **complete intersection**. In this case, V is generated by the smallest number of polynomials that can produce a variety of dimension d in $\mathbf{A}^n$ or $\mathbf{P}^n$.

Fix a point a in V. Let an **affine neighborhood** of a be a set U such that (1) $U \subseteq U_i \subseteq \mathbf{P}^n$ for some standard affine set U_i; (2) U contains a; and (3) U is closed in U_i. The following properties are equivalent (proof omitted):

1. There exists an affine neighborhood U of a such that the ideal of polynomials in $K[U]$ that vanish on $U \cap V$ is generated by $n-d$ polynomials.
2. There exists a d-dimensional variety $W \subseteq \mathbf{P}^n$ such that $a \notin W$ and $V \cup W$ is a complete intersection.

If these properties hold, then we say that V is a **local complete intersection** at a.

V is a **local complete intersection** if it is a local complete intersection at every point a in V.

14. Hilbert Polynomials

Let $V \subseteq \mathbf{P}^n$ be a projective variety, and let $K[V] = K[Z]/I(V)$ (§ 7.2). For each natural number m, let $K[V]_m$ denote the set of homogeneous polynomials of $K[V]$ with degree m, and let dim $K[V]_m$ denote the dimension of $K[V]_m$ as a vector space over V.

1. The **Hilbert function** of V is the function $h_V\colon \mathbf{N} \to \mathbf{N}$ given by

$$h_V(m) = \dim\ K[V]_m.$$

   Because $\dim\ K[V]_m = \dim\ K[Z]_m - \dim\ I(V)_m$, we say that $h_V(m)$ is the **codimension** in $K[Z]_m$ of the space $I(V)_m$ of homogeneous polynomials of degree m that vanish on V.

2. There exist a polynomial $p_V(Z)$ and a natural number M such that for all $m \geq M$, $h_V(m) = p_V(m)$. The polynomial p_V is called the **Hilbert polynomial** of the projective variety V. The degree of p_V is the dimension of V (proof omitted).

15. Tangent Spaces; Smooth and Singular Points and Varieties

In this section we define the concept of a tangent space to a variety. We also define the related concepts of smooth and singular points of a variety.

15.1. The Zariski Tangent Space at a Point

Let V be a variety, and fix a point a of V. Let $\mathbf{O}_a(V)$ be the local ring of V at a (§ 7), and let $\mathbf{m}_a$ be the maximal ideal in $\mathbf{O}_a(V)$ of polynomials that vanish at a.

1. Let F_a be the field $\mathbf{O}_a(V)/\mathbf{m}_a$. The **Zariski cotangent space** to V at a is the F_a-vector space defined as follows:

$$T_a^*(V) = \mathbf{m}_a/\mathbf{m}_a^2.$$

   $T_a^*(V)$ is a F_a-vector space because it is an $\mathbf{O}_a(V)$-module that is annihilated by $\mathbf{m}_a$.

2. The **Zariski tangent space** to V at a, denoted $T_a(V)$, is the dual space of $T_a^*(V)$, i.e., the space of linear maps $\lambda\colon T_a^*(V) \to F_a$:

$$T_a(V) = (T_a^*(V))^*.$$

Let V and W be varieties, and let $\psi\colon V \to W$ be a regular map. For each point a in V, ψ induces the following maps:

1. A map $\psi^*\colon \mathbf{O}(W)_{\psi(a)} \to \mathbf{O}_a(V)$ given by $\psi^*(p) = p \circ \psi$.
2. A map $\theta\colon T_a(V) \to T_{\psi(a)}(W)$ given by $\theta(\lambda) = \lambda \circ \psi^*$, where $\lambda\colon T_a^*(V) \to F_a$ is an element of $T_a(V) = (T_a^*(V))^*$.

In particular, when V is an affine variety, the embedding $\psi\colon V \to \mathbf{A}^n$ induces an embedding

$$\theta\colon T_a(V) \to T_{\psi(a)}(\mathbf{A}^n) = K^n.$$

In this case the embedding θ has the following alternative and equivalent definition (proof omitted):

1. Fix a family $\{p_i\}_{i \in [1,m]}$ of polynomials in $K[z]$ that generate the ideal $I(V)$. For each i in $[1, m]$, p_i defines a differentiable function $f\colon K^n \to K$ given by $f_i(v) = p_i(v)$. Let $L(K^n, K)$ denote the space of linear maps $\lambda\colon K^n \to K$. The derivative Df_i is the map from K^n to $L(K^n, K)$ given by

$$Df_i(z)(h) = \sum_{j=1}^{n} D_j f_i(z)(h_j),$$

   where $D_j f_i(z)\colon K \to K$ denotes the jth partial derivative of f_i (i.e., the derivative of f_i treated as a function of the single variable z_j), and h_j is the jth coordinate of h. See § 6.2 of my paper *The General Derivative*.

2. The Zariski tangent space $T_a(V)$ is the set of all vectors h in K^n such that

$$Df_i(a)(h) = 0$$

   for all i in $[1, m]$.

Note the following:

1. $T_a(V) = \bigcap_{i \in [1,m]} T_a(V_i)$, where V_i is the zero set of f_i.
2. Fix an index i in $[1,m]$ and a point a in V. If $Df_i(a) \neq 0$, then $T_a(V_i)$ is the ordinary tangent space of the graph of f_i from calculus and differential geometry. For example, in $\mathbf{A}^2$, let $f_i(z) = z_1^2 - z_2$. Then

$$Df_i(z)(h) = 2z_1h_1 - h_2 = (2z_1, -1) \cdot h,$$

so $T_a(V_i)$ is the space of vectors $h = (h_1, h_2)$ that are perpendicular to $(2a_1, -1)$.

15.2. Smooth and Singular Points

Let V be a variety in $\mathbf{A}^n$ or $\mathbf{P}^n$ of pure dimension d, and fix a point a of V.

1. The Zariski tangent space $T_a(V)$ is a subspace of K^n and is isomorphic to K^m, for some m with $d \leq m \leq n$ (§ 13) (proof omitted).
2. If $m = d$, then we say that a is a **smooth point** of V. Otherwise we say that a is a **singular point** of V.

Let V be a variety of pure dimension d.

1. The set of singular points of V is a subvariety of V (proof omitted). We denote this subvariety V_{sing}.
2. The set of smooth points of V is an open and dense subset of V (proof omitted). We denote this set V_{sm}.
3. If V has no singular points, then we say that V is a **smooth variety**. Otherwise V is a **singular variety**.
4. Assume that V is irreducible. If the field K has characteristic zero, then there exists a smooth irreducible variety W and a regular birational map $\phi\colon W \to V$ (proof omitted). If V is singular, then the map ϕ is called a **resolution of singularities** of V.

15.3. The Affine Tangent Plane at a Point

Let $V \subseteq \mathbf{A}^n$ be an affine variety. The **affine tangent plane** of V at a, denoted $\mathbf{T}_a(V)$, is the set of points b in $\mathbf{A}^n$ such that $b - a$ lies in the Zariski tangent space $T_a(V)$, considered as a subspace of $\mathbf{A}^n$. If V has pure dimension d and a is a smooth point of V, then $\mathbf{T}_a(V)$ is a d-plane in $\mathbf{A}^n$ and is the image of the d-plane $T_a(V)$ under the translation that takes the origin to a.

For example, let $K = \mathbf{C}$, let $n = 2$, let V be the zero locus of $z_1^2 - z_2$, and let $a = (1, 1)$. Then

1. $T_a(V)$ is the line through $(0, 0)$ perpendicular to the line through $(0, 0)$ and $(2, -1)$.
2. $\mathbf{T}_a(V)$ is the line through $(1, 1)$ perpendicular to the line through $(1, 1)$ and $(3, 0)$.

15.4. The Projective Tangent Plane at a Point

In this section, $V \subseteq \mathbf{P}^n$ is a projective variety, and a is a point of V with homogeneous coordinates $[a_i]$.

Fix a family $\{P_i\}_{i \in [1,m]}$ of homogeneous polynomials in $K[Z]$ that generate the ideal $I(V)$, and let $\{F_i\}_{i \in [1,m]}$ be the corresponding family of functions from K^{n+1} to K. The **projective tangent plane** of V at a, denoted $\mathbf{T}_a(V)$, is the set of points b in $\mathbf{P}^n$ with homogeneous coordinates $[b_i]$ such that

$$DF_i(a)(b) = 0$$

for all i in $[1,m]$. Note that $\mathbf{T}_a(V)$ is a well-defined subset of $\mathbf{P}^n$, because if $F_i(Z)$ is a homogeneous polynomial of degree d, then by the rule for differentiating polynomials

$$DF_i(X)(Y) = \sum_{j=0}^{n} DF_i(X)Y_j$$

is a bihomogeneous polynomial of degree $d - 1$ in X and degree 1 in Y.

Let U_k be any of the standard affine open sets $U_k \subseteq \mathbf{P}^n$ containing a. Then

$$\mathbf{T}_a(V \cap U_k) = \mathbf{T}_a(V) \cap U_k,$$

where $\mathbf{T}_a$ on the left denotes the affine tangent plane (§ 15.3). Indeed, suppose $k = 0$, and assume $a_0 = 1$. Choose any i in $[1,m]$ and let $F = F_i$.

1. By the Euler relation for homogeneous polynomials we have
$$\sum_{j=0}^{n} D_j F(Z)\ Z_j = d \cdot F(Z),$$
where d is the degree of F.
2. Because $F(a) = 0$, we have
$$\sum_{j=0}^{n} D_j F(a)\ a_j Z_0 = d \cdot F(a)\ Z_0 = 0.$$
Therefore
$$D_0 F(a)\ Z_0 = D_0 F(a)\ a_0 Z_0 = -\sum_{j=1}^{n} D_j F(a)\ a_j Z_0. \tag{4}$$
3. By definition, $\mathbf{T}_a(V)$ is the set of all b in $\mathbf{P}^n$ such that
$$\sum_{j=0}^{n} D_j F(a) b_j = 0. \tag{5}$$
By (4), the left-hand side of (5) is
$$\begin{aligned}(\sum_{j=1}^{n} D_j F(a) b_j) + D_0 F(a) b_0 &= \sum_{j=1}^{n} D_j F(a) b_j - \sum_{j=1}^{n} D_j F(a)\ a_j b_0 \\ &= \sum_{j=1}^{n} D_j F(a)(b_j - a_j b_0). \qquad (6)\end{aligned}$$
4. When intersecting with U_0, we may assume $b_0 = 1$. Therefore, by (5) and (6), $\mathbf{T}_a(V) \cap U_0$ contains all and only the points satisfying
$$\sum_{j=1}^{n} D_j F(a)(b_j - a_j) = 0. \tag{7}$$
These are exactly the points satisfying the definition of $\mathbf{T}_a(V \cap U_0)$ (§ 15.3).

Let $W \subseteq \mathbf{A}^{n+1}$ be the zero locus of the ideal $I(V)$, considered as a set of polynomials $p(z_0, \ldots, z_n)$. Then $\mathbf{T}_a(V)$ is the subspace of $\mathbf{P}^n$ corresponding to $T_a(W)$ (proof omitted).

16. The Degree of a Variety

Let V and W be linear subspaces of $\mathbf{P}^n$ (§ 4.4).

1. The **span** of V and W, denoted $\overline{V, W}$, is the set of all points $[a + b]$, where $[a]$ is a point of V, $[b]$ is a point of W, and $(a + b)_i = a_i + b_i$. Equivalently, $\overline{V, W}$ is the smallest linear subspace of $\mathbf{P}^n$ that contains both V and W.
2. The dimension of $\overline{V, W}$ is the minimum of n and m, where $m = \dim V + \dim W - \dim (W \cap V)$ (proof omitted).
3. If V is a point a and W is a different point b, the span $\overline{a, b}$ is the unique line in $\mathbf{P}^n$ passing through a and b.
4. Assume that V has dimension m and that W is a point a in $\mathbf{P}^n - V$. The span $\overline{a, V}$ is called a **cone**. It is the union of all the lines $\overline{a, b}$ for which b is a point of V. It is an $(m + 1)$-plane in $\mathbf{P}^n$.

Let m be a natural number in $[0, n)$. Let V and W be subvarieties of $\mathbf{P}^n$ such that V is isomorphic to $\mathbf{P}^m$, W is isomorphic to $\mathbf{P}^{n-m-1}$, and $V \cap W = \varnothing$.

1. The **projection** of $\mathbf{P}^n - V$ to W is the map $\pi_V : \mathbf{P}^n - V \to W$ that sends a point a of $\mathbf{P}^n - V$ to the intersection of W with the $(m + 1)$-plane $\overline{a, V}$. This map is well-defined because the intersection is a single point (proof omitted). If $X \subseteq \mathbf{P}^n - V$ is a projective variety in $\mathbf{P}^n$, then $\pi_V(X)$ is a projective variety in $W = \mathbf{P}^{n-m-1}$ (proof omitted).

2. Up to projective equivalence (§ 8.2), we may choose V and W as follows:
   - The points of V are the points of $\mathbf{P}^n$ whose last $n-m$ homogeneous coordinates are zero.
   - The points of W are the points of $\mathbf{P}^n$ whose first $m+1$ homogeneous coordinates are zero.

   In this case π_V is the standard coordinatewise projection map that takes $[a_0, \ldots, a_m, a_{m+1}, \ldots, a_n]$ to $[a_{m+1}, \ldots, a_n]$ (proof omitted). This map is well-defined on $\mathbf{P}^n - V$, because on that domain one of the coordinates a_{m+1} through a_n must be nonzero.

Let $V \subseteq \mathbf{P}^n$ be an irreducible variety of dimension m. We define the **degree** of V in the following ways, all of which are equivalent (proof omitted).

1. Let W be a general $(n-m-2)$-plane in $\mathbf{P}^n$. Then
   a. $\pi_W(V)$ is an irreducible hypersurface in $\mathbf{P}^{m+1}$ (proof omitted).
   b. For any W, the degree of the irreducible homogeneous polynomial $P(Z)$ defining $\pi_W(V)$ is the same natural number d (proof omitted).

   The degree of V is d.
2. Let X be a general $(n-m-1)$-plane in $\mathbf{P}^n$. Then
   a. π_X defines a surjective map from V to $\mathbf{P}^m$ (proof omitted).
   b. For the general point a of $\mathbf{P}^m$, the set $\pi_X^{-1}(a)$ contains d points, for some natural number d (proof omitted).

   The degree of V is d.
3. Let Y be a general $(n-m)$-plane in $\mathbf{P}^n$. Then V intersects Y in d points, for some natural number d that does not depend on Y (proof omitted). The degree of V is d.
4. Suppose V has dimension m, and let p_V be the Hilbert polynomial of V (§ 14). Then the degree of V is $m!$ times the leading coefficient of p_V.

We write deg V to denote the degree of V.

Let V and W be quasi-projective varieties in $\mathbf{P}^n$.

1. Let a be a point of $V \cap W$. V and W **intersect transversely at** a if V and W are smooth at a (§ 15.2) and the span of $\mathbf{T}_a(V)$ and $\mathbf{T}_a(W)$ is $\mathbf{P}^n$ (§ 15.4). For example, projecting from $\mathbf{P}^2$ onto $\mathbf{A}^2$, let V be the curve $y = x^2$, and let W be the line $y = 1$. Then V and W intersect transversely at $(1, 1)$, because the tangent line to V and $(1, 1)$ and the line $y = 1$ span $\mathbf{A}^2$. On the other hand, if W is the line $y = 0$, then V and W do not intersect transversely at $(0, 0)$, because the tangent line to V at $(0, 0)$ and the line $y = 0$ span a space of dimension one.
2. Let $\{X_i\}$ be the irreducible components of $V \cap W$ (§ 3.3). V and W **intersect generically transversely** if, for each i, V and W intersect transversely at the general point a of X_i.
3. Assume that V has pure dimension d_V, W has pure dimension d_W, $d_V + d_W \geq n$, and V and W intersect generically transversely. Then

   $$\deg\,(V \cap W) = (\deg V)(\deg W).$$

   This statement is called **Bézout's Theorem** (proof omitted).

Let V and W be quasi-projective varieties in $\mathbf{P}^n$, each with pure dimension.

1. V and W **intersect properly** if

   $$\operatorname{codim}\,(V \cap W) = \operatorname{codim} V + \operatorname{codim} W$$

   or equivalently

   $$\dim\,(V \cap W) = \dim V + \dim W - n.$$

2. Assume that V and W intersect properly, and let $\{X_i\}$ be the irreducible components of $V \cap W$. There exists a family $\{m_i\}$ of natural numbers such that the following properties hold (proof omitted):
   a. $m_i \geq 1$, and $m_i = 1$ if and only if V and W intersect transversely at a general point of X_i.

b. $(\deg V)(\deg W) = \sum_i m_i \deg X_i$.

For each i, m_i is called the **intersection multiplicity** or **intersection number** of V and W along X_i.[2]

17. Parameter Spaces

In this section, $B \subseteq \mathbf{P}^m$ is a variety, $\pi\colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^m$ is the map $(b, c) \mapsto b$, and $\phi\colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N = \mathbf{P}^{mn+m+n}$ is the Segre map (§ 9.2).

Let $V \subseteq \phi(B \times \mathbf{P}^n)$ be a variety. V induces a family of subsets $\{V_b\}_{b \in B}$ of $\mathbf{P}^N$ according to the rule $V_b = \phi(\pi^{-1}(b))$.

1. If each set $V_b \subseteq \mathbf{P}^N$ is a variety, then we say that $F = (B, V)$ is an **algebraic family of varieties in $\mathbf{P}^N$**. We call B the **base** of F, and we call V the **total space** of F.
2. Let $F = (B, V)$ be an algebraic family of varieties in $\mathbf{P}^N$. If V is closed, then we say that F is a **closed algebraic family**. In this case, F has the following properties:
   a. Each V_b is the intersection of the closed sets V and $\phi(\{b\} \times \mathbf{P}^n)$, so each V_b is a projective variety.
   b. If B is an affine variety, then F is defined by a set of homogeneous polynomials in $n+1$ variables whose coordinates are polynomials in m variables.
   c. If B is a projective variety, then F is defined by a set of homogeneous polynomials in $n+1$ variables whose coordinates are homogeneous polynomials in $m+1$ variables.

We now define the concept of a reduced algebraic family of varieties.[3] Let $F = (B, V)$ be an algebraic family of varieties in $\mathbf{P}^N$ with base $B \subseteq \mathbf{P}^m$ and total space $V \subseteq \mathbf{P}^N$.

1. For any point b of B and any point v of $V_b \subseteq V$, let $f(b, v)\colon \mathbf{O}_b(B) \to \mathbf{O}_v(V)$ be the map $g \mapsto g \circ \pi$, where g is a regular function germ (§ 7.2). This map is well-defined because
   a. For any polynomials p and q in $K[z]$, if $p(b) = q(b)$ for all b in B then $(p \circ \pi)(v) = (q \circ \pi)(v)$ for all v in V, so $p \sim q$ in $K[B]$ implies $f(b, v)(p) \sim f(b, v)(q)$ in $K[V]$.
   b. For any polynomial q in $K[B]$, if $q(b) \neq 0$ then $(q \circ \pi)(v) \neq 0$, so for any function germ $g = p/q$ in $\mathbf{O}_b(B)$, $f(b, v)(p/q) = (p \circ \pi)/(q \circ \pi)$ is an element of $\mathbf{O}_v(V)$.
2. Fix point b of B and a point v of V_b. Let $\mathbf{m}_b \subseteq \mathbf{O}_b(B)$ be the maximal ideal of functions vanishing at b. If the set $f(b, v)(\mathbf{m}_b)$ generates the ideal of V_b in $\mathbf{O}_v(V)$, then we say that the algebraic family F is **reduced** with respect to the pair (b, v).
3. If, for all points b in B and all points v in V_b, F is reduced with respect to (b, v), then we say that F is **reduced**.
4. If, for all points b in B and the generic point v of V_b, F is reduced with respect to (b, v), then we say that F is **generically reduced**.

Now we define what it means for the variety V to be a parameter space for the family $F = (B, V)$. Fix a variety $C \subseteq \mathbf{P}^{m_C}$ for some natural number m_C.

1. Let $R(F, C)$ be the set of regular maps from C to B.
2. Let $\mathbf{F}(F, C)$ be the set of families $\{X_c\}_{c \in C}$ of varieties in $\mathbf{P}^n$ parameterized by C such that for each c in C, $X_c = V_b$ for some b in B. Here equality means set equality as subsets of $\mathbf{P}^n$.
3. For any regular map $\psi\colon C \to B$, define a family of varieties $G(F, C, \psi) = \{X(F, C, \psi)_c\}_{c \in C}$ according to the rule $X(F, C, \psi)_c = V_{\psi(c)}$.
4. Let $\Phi(F, C)\colon R(F, C) \to \mathbf{F}(F, C)$ be the map that takes ψ to $G(F, C, \psi)$.

Let $\mathbf{F}_{\text{cr}}(F, C)$ be the subset of $\mathbf{F}(F, C)$ consisting of all closed reduced algebraic families. If, for every variety C, $\Phi(F, C)$ is a bijection between $R(F, C)$ and $\mathbf{F}_{\text{cr}}(F, C)$, then we say that the variety V is a **parameter space** for F.

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[2] If V and W are affine plane curves, then m_i has a constructive definition that is relatively straightforward. See [Fulton 2008], § 3.3. If V and W are general quasi-projective varieties, then the constructive definition of m_i requires the machinery of **intersection theory**, which is a subtopic of modern algebraic geometry. See [Fulton 1984].

[3] Following [Harris 1992], we introduce this concept in order to define parameter spaces in the language of varieties. Modern algebraic geometry uses the alternative concept of a "flat family of schemes." See [Harris 1992], p. 267.

Let $\mathbf{F}_{\mathrm{cgr}}(F, C)$ be the subset of $\mathbf{F}(F, C)$ consisting of all closed, generically reduced algebraic families. If, for every variety C, $\Phi(F, C)$ is a bijection between $R(F, C)$ and $\mathbf{F}_{\mathrm{cgr}}(F, C)$, then we say that the variety V is a **cycle parameter space** for F.

Notice that each variety V_b in the family $F = \{V_b\}$ has the form $\phi(\{b\} \times W_b)$, for some set $W_b \subseteq \mathbf{P}^n$. It is often the case that $\{W_b\}_{b \in B}$ is also a family of varieties, and it is useful to think of V as a parameter space for this family.

18. Moduli Spaces

In this section, we define the concept of a **moduli space**. Like a parameter space (§ 17), a moduli space is a family of varieties parameterized by another variety, with some additional structure. Unlike a parameter space, the varieties need not be embedded in a fixed space $\mathbf{P}^N$.

Fix a variety B. Let V be a variety, and let $\pi: V \to B$ be a map. The map π induces a family of sets $\{V_b\}_{b \in B}$ according to the rule $V_b = \pi^{-1}(b)$.

1. If each V_b is a variety, then we may form the family $\{[V_b]\}_{b \in B}$ of isomorphism classes of varieties (§ 8.3). In this case we say that $F = (B, V, \pi)$ is an **algebraic family of abstract varieties**. Again we call V the **total space** of F, and we call B the **base** of F.
2. Let F be an algebraic family of abstract varieties. We say that F is **reduced** if it is reduced according to the definition given in § 17 for algebraic families of varieties in $\mathbf{P}^N$. This definition is equally valid for families of abstract varieties.

Fix a variety B, and let $S = \{X_b\}_{b \in B}$ be a standard (i.e., not necessarily an algebraic) family of abstract varieties.

1. For any variety C, define the following:
    a. Let $\mathbf{F}(S, C)$ be the set of algebraic families of abstract varieties (C, V, π) such that for each c in C, we have $[V_c] = X_b$ for some b in B. Here equality means identity of abstract varieties (i.e., isomorphism classes).
    b. For any family $F = (C, V, \pi)$ in $\mathbf{F}(S, C)$, let $\psi(C, F): C \to B$ be the map $c \mapsto b$, where $[V_c] = X_b$.
2. B is a **coarse moduli space** for the family S if the following hold:
    a. For any variety C and any reduced family F in $\mathbf{F}(S, C)$, $\psi(C, F)$ is a regular map.
    b. B is unique up to isomorphism in the following sense. Suppose the following:
        i. Item a holds for another variety B' and another family $S' = \{X'_b\}_{b \in B'}$.
        ii. $\eta: B' \to B$ is a regular bijection.
        iii. For every variety C and for every family F in $\mathbf{F}(S, C)$ there is a family F' in $\mathbf{F}(S', C)$ such that $\psi(C, F) = \eta \circ \psi(C, F')$.

        Then B' is isomorphic to B.

Notice that unlike a parameter space, a coarse moduli space is not the total space of an algebraic family. If there exists an algebraic family $F = (B, V, \pi)$ such that $\{[V_b]\}_{b \in B} = S$, then we say that F is a **tautological family** over B. The variety V in the tautological family is then analogous to the parameter space V of an algebraic family in $\mathbf{P}^N$.

Suppose there exists a tautological family F over B.

1. For any variety C, let $R(S, C)$ be the set of regular maps from C to B, and let $\mathbf{F}_{\mathrm{r}}(S, C)$ be the subset of $\mathbf{F}(S, C)$ consisting of reduced families.
2. Let $\Phi(S, C): \mathbf{F}_{\mathrm{r}}(S, C) \to R(S, C)$ be the map that takes a family G to the regular map $\psi(C, G)$. If Φ is a bijection, then we say that F is a **universal family**. In this case, we say that B is a **fine moduli space** for the family S.

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