Definitions for Category Theory

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This document defines concepts used in the area of mathematics known as **category theory**. Category theory is the study of **categories**, i.e., collections of objects and "arrows" (usually mappings) between them. Examples of categories described as objects (with arrows) include sets (with functions), groups (with homomorphisms), and vector spaces (with linear maps). In fact most areas of mathematics use categories in one way or another, since they all involve mappings between structures. The language of categories provides a way to unify all these areas of mathematics as specific instances of the same general theory.

The motivation for this document is the same as stated in my paper *Definitions for Commutative Algebra*: it is not so easy to remember all these definitions, and it is not so easy to extract them from the textbooks in which they are embedded. A little summarizing can go a long way here.

Note on typography: In discussions of category theory, it is customary to omit parentheses as much as possible in function applications. For example, whereas in algebra or analysis we might write $f(a)$ for the mapping f applied to the value a, in category theory we usually write $f\ a$. We generally follow that practice here.

Sometimes we will string three or more symbols together without parentheses, e.g., $a\ b\ c$. Again this is common practice. We will assume left associativity unless otherwise specified, e.g., $a\ b\ c$ means $(a\ b)\ c$.

1. Set Theory

The standard development of category theory is based on set theory. So we begin with set theory. We use ZFC, i.e., Zermelo-Fraenkel set theory with the axiom of choice. For an introduction to ZFC, see my paper *Zermelo-Fraenkel Set Theory*.

A **universe** is a set U with the following properties:

1. For every nonempty set $S \in U$, every element of S is an element of U. That is, U is transitively closed with respect to membership (a member of a member is a member).
2. Every axiom of ZFC holds after making the following replacements:
   a. Replace all formulas $\forall x\ p$ with $\forall x(x \in U \Rightarrow p)$.
   b. Replace all formulas $\exists x\ p$ with $\exists x(x \in U \wedge p)$.

Intuitively, U is a collection of sets that (1) is rich enough to support all the operations provided by ZFC; and (2) is itself a set. ZFC itself does not provide any such set; in particular there is no "set of all sets" in ZFC.

We assume the existence of a universe U that contains all the objects we ever need for ordinary (i.e., non-category-theoretic) mathematics. For historical reasons, we call the members of U **small** objects, although "ordinary mathematical objects" would also be apt. If a ZFC set S is not small (i.e., is not a member of U), then we will call S a **large** set. In particular, U is a large set, since by the Axiom of Regularity we have $U \notin U$. The large sets are for category theory only; all non-category-theoretic mathematics uses small sets.

With this division of sets into small and large, we can construct the set of all small (i.e., ordinary mathematical) objects of a particular type. For example, we can construct the set of all small sets, or the set of all small groups. These are just all the sets or groups that are members of U. This operation is valid by the Axiom Schema of Specification. In general, the set of all small objects of a particular type is a large set. We will use large sets to define categories.[1]

[1] An alternate foundation for category theory uses Von Neumann-Bernays-Gödel set theory (NBG). This theory distinguishes between sets and **proper classes** which are not sets. For example, in NBG, the collection of all sets is a proper class. We can reconcile the two foundations by thinking of the universe set U in the ZFC-based theory as a proper class in the NBG-based theory. The NBG-based theory provides less flexibility when constructing large categories: for example, one can take the power set of the large set U, but it is not possible to take the "power class"

2. Categories

A **category** is a tuple $(O, A, I, \circ)$, defined as follows:

1. O is a set of **objects**.
2. A is a set of **arrows** $a \xrightarrow{f} b$. Here f is a label that uniquely identifies the arrow, and a and b are members of O. We also write an arrow $a \xrightarrow{f} b$ by writing $f \colon a \to b$, or by writing f if the objects a and b are implied. We write $a \to b$ to denote any arrow from a to b, without specifying the label.
3. $I \colon O \to A$ is a mapping that assigns to each object a in O an **identity arrow** denoted $a \xrightarrow{\mathrm{id}_a} a$ or $a \xrightarrow{1_a} a$.
4. An ordered pair of arrows $(a \xrightarrow{f} b, c \xrightarrow{g} d)$ is called **composable** if $b = c$. Let $P \subseteq A \times A$ be the set of all pairs of composable arrows. Then $\circ \colon P \to A$ is a mapping that assigns to each pair $(a \xrightarrow{f} b, b \xrightarrow{g} c) \in P$ a **composite arrow** $a \xrightarrow{f} b \xrightarrow{g} c$ or $a \xrightarrow{g \circ f} c$, such that the following properties hold:
   a. For all pairs of composable pairs of arrows $((f, g), (g, h))$, we have associativity, i.e.,
   $$h \circ (g \circ f) = (h \circ g) \circ f.$$
   In light of this identity, we can omit the parentheses and write $h \circ g \circ f$.
   b. For all arrows $a \xrightarrow{f} b$, we have
   $$f \circ 1_a = 1_b \circ f = f.$$

Note that when writing a composite $a \xrightarrow{f} b \xrightarrow{g} c$ with arrows, the order of arrows is left to right, but in the open circle notation $g \circ f$, the order is right to left. The "backwards" ordering in the circle notation is motivated by the order of application in mappings of sets, e.g., $(g \circ f)\ x = g\ (f\ x)$.

In item 2, the object a is called the **domain** of f and written $\mathrm{dom}\ f$. The object b is called the **codomain** of f and written $\mathrm{cod}\ f$. We say that an arrow goes from its domain to its codomain. An ordered pair of arrows (f, g) is composable if and only if the codomain of f equals the domain of g.

Distinct labels represent distinct arrows, even with the same domain and codomain. For example, $a \xrightarrow{f} b$ and $a \xrightarrow{g} b$ are distinct arrows if f and g are distinct labels.

Categories abound in mathematics. For example, let V be a set containing some sets and all mappings between them. We define the category $\mathbf{Set}_V$ as follows:

1. O is the set of all sets in V.
2. A is the set of all mappings in V between sets in V. An arrow $a \xrightarrow{f} b$ is a mapping $f \colon a \to b$.
3. I assigns to each set S in O the identity mapping on S.
4. $\circ$ is ordinary composition of mappings between sets, i.e.,
   $$g \circ f = x \mapsto g\ (f\ x).$$

When $V = U$ (§ 1), $\mathbf{Set}_V = \mathbf{Set}$, the category of all small sets and small mappings between them.[2]

We define the categories $\mathbf{Grp}_V$ and $\mathbf{Grp}$ similarly, replacing "set" with "group" and "mapping" with "group homomorphism." Similarly we can define $\mathbf{Rng}_V$ and $\mathbf{Rng}$ (rings with ring homomorphisms) and R-$\mathbf{Mod}_V$ and R-$\mathbf{Mod}$ (modules over a commutative ring R with R-module homomorphisms). Thus the concept of a category captures what is common among sets, groups, rings, R-modules, and many other mathematical structures.

We often omit the symbol $\circ$ when writing a composition of arrows. For example, instead of $g \circ f$ we may write $g\ f$. In this notation, when the composition of g and f is a map applied to the element a, we may write $g\ f\ a$ without ambiguity. By definition $(g \circ f)\ a = (g\ f)\ a = g\ (f\ a)$.

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of a proper class. See [Mac Lane 1998], pp. 23–24.

[2] [Mac Lane 1998] refers to $\mathbf{Set}_V$ as **Ens**. Apparently this notation comes from *ensemble*, the French word for set.

Fix a category $C = (O, A, I, \circ)$. A category $C' = (O', A', I', \circ')$ is a **subcategory** of C if each of the elements O', A', I', and $\circ'$ is included in the corresponding element of C (this means set inclusion, treating mappings as sets of ordered pairs).

A **discrete** category is a category in which the arrow set A contains only the identity arrows for the objects in O. Given any discrete category C, there is a corresponding set of objects O; and given any set of objects S we can make it into the set O of a discrete category by adding all the identity arrows. Thus discrete categories correspond to sets. For example, the category consisting of objects 1, 2, and 3 with an identity arrow for each object corresponds to the set $\{1, 2, 3\}$.

When C is a category, we will write $c \in C$ to mean "c is an object of C," i.e., $C = (O, A, I, \circ)$ and $c \in O$.

3. Diagrams

Fix a category C. A **diagram** in C is a written representation of a collection of objects and arrows of C, in which the objects appear as letters or symbols, each arrow goes from its domain object to its codomain object, and the arrows appear with or without labels. For example, the representation $a \xrightarrow{f} b$ of the arrow f from a to b is a diagram. We may also write $a \to b$, omitting the label f. As another example, Figure 1 shows objects a, b, and c and arrows $a \xrightarrow{f} b$, $b \xrightarrow{g} c$, and $a \xrightarrow{h} c$.

$$
\begin{tikzcd}
a \arrow[r, "h"] \arrow[d, "f"'] & c \\
b \arrow[ur, "g"'] &
\end{tikzcd}
$$

Figure 1: A diagram.

Let D be a diagram. We say that D is **commutative** (or **commutes**) if every path through the diagram with the same endpoints, treating successive arrows as composition, represents the same map. For example, the diagram shown in Figure 1 commutes if $h = g\ f$.

4. Hom Sets

Fix a category C and objects a and b of C. We write $\hom_C(a, b)$ or $C(a, b)$ to denote the set of all arrows $a \xrightarrow{f} b$ from a to b in C. We call this set a **hom set**. (The terminology comes from abstract algebra; "hom" stands for homomorphism. A better name would be an arrow set.) We also denote this set $\hom(a, b)$ when the category C is clear.

We may represent the associativity of composition of arrows as the commutative diagram of hom sets shown in Figure 2.

$$
\begin{tikzcd}
\hom(c, d) \times \hom(b, c) \times \hom(a, b) \arrow[r, "{(h, g, f) \mapsto (h \circ g, f)}"] \arrow[d, "{(h, g, f) \mapsto (h, g \circ f)}"'] & \hom(b, d) \times \hom(a, b) \arrow[d, "{(h \circ g, f) \mapsto (h \circ g) \circ f}"] \\
\hom(c, d) \times \hom(a, c) \arrow[r, "{(h, g \circ f) \mapsto h \circ (g \circ f)}"'] & \hom(a, d)
\end{tikzcd}
$$

Figure 2: A commutative diagram representing the associativity of composition of arrows.

In some categories, hom sets have additional structure. For example, in the category R-**Mod** of modules over the commutative ring R, each hom set is itself an R-module.

Fix a category C and arrows $a' \xrightarrow{f} a$ and $b \xrightarrow{g} b'$ of C.

1. We write $\hom(f, g)$ to denote the mapping from $\hom(a, b)$ to $\hom(a', b')$ given by
$$h \mapsto g\ h\ f.$$
We say that hom is **contravariant** in its first argument, because it reverses the positions of a and a'. We take up this idea further below, in the section on functors.
2. We write $\hom(f, b)$ in the case where $b' = b$ and $g = 1_b$. $\hom(f, b)$ is the mapping $h \mapsto h\ f$ that says, "Given a homomorphism h, construct a new homomorphism $\hom(f, b)\ h$ by composing h with f on the right."
3. We write $\hom(a, g)$ in the case where $a' = a$ and $f = 1_a$. $\hom(a, g)$ is the mapping $h \mapsto g\ h$ that says, "Given a homomorphism h, construct a new homomorphism $\hom(a, g)\ h$ by composing h with g on the left."

Many authors write f^* for $\hom(f, b)$ and g_* for $\hom(a, g)$.

5. Properties of Objects and Arrows

Fix a category C and an arrow $a \xrightarrow{f} b$ in C.

1. f is **monic** or **left cancellable** if for every object c and pair of arrows $c \xrightarrow{g} a$ and $c \xrightarrow{h} a$ such that $f\ g = f\ h$, we have $g = h$. In **Set**, the monic arrows are the injective maps.
2. f is **epi** or **right cancellable** if for every object c and pair of arrows $b \xrightarrow{g} c$ and $b \xrightarrow{h} c$ such that $g\ f = h\ f$, we have $g = h$. In **Set**, the monic arrows are the surjective maps.
3. A **right inverse** for f (also called a **section** of f) is an arrow $b \xrightarrow{g} a$ with $f\ g = 1_b$. If f has a right inverse, then f is epi. Indeed, if $h\ f = h'\ f$, then $h\ f\ g = h'\ f\ g$, so $h\ 1_b = h'\ 1_b$, so $h = h'$.
4. A **left inverse** for f (also called a **retraction** for f) is an arrow $b \xrightarrow{g} a$ with $g\ f = 1_a$. If f has a left inverse, then f is monic. Indeed, if $f\ h = f\ h'$, then $g\ f\ h = g\ f\ h'$, so $1_a\ h = 1_a\ h'$, so $h = h'$.
5. An **inverse** for f is an arrow $b \xrightarrow{f^{-1}} a$ that is both a right inverse and a left inverse for f. Such an arrow, if it exists, is unique. If f^{-1} exists, then we say that f is **invertible**. We also say that f is an **isomorphism** and that a and b are **isomorphic**, and we write $a \cong b$. If (f, g) is a composable pair of invertible arrows, then $(g\ f)^{-1} = f^{-1}\ g^{-1}$.

Fix a category C and an arrow $a \xrightarrow{f} a$ in C.

1. f is **idempotent** $f\ f = f$.
2. f **splits** if it is idempotent and there exist arrows $a \xrightarrow{g} b$ and $b \xrightarrow{h} a$ such that $f = h\ g$ and $g\ h = 1_b$. In this case, g has a right inverse and so is epi, and we say that g is a **split epi**. Similarly, h has a left inverse and so is monic, and we say that h is a **split monic**.

The concept of an arrow that splits generalizes the concept of a split exact sequence in homological algebra. Fix a ring R, let B and C be R-modules (i.e., objects in the category R-Mod), and let $A = B \oplus C$. A **split exact sequence** is a diagram of the form

$$0 \to B \xrightarrow{h} A = B \oplus C \xrightarrow{h'} C \to 0$$

where $h\ b = (b, 0)$ and $h'\ (b, c) = c$. Setting $g\colon A \to B = (b, c) \mapsto b$ and $f\colon B \to B = h\ g = (b, c) \mapsto (b, 0)$, we see that f is idempotent, $f = h\ g$, and $g\ h = b \mapsto b = 1_B$. Therefore f splits in the category theoretic sense. Similarly, if we set $g'\colon C \to A = c \mapsto (0, c)$, then $f' = g'\ h'$ splits.

Fix a category C and an object a in C.

1. a is **initial** if for every object b in C, there is exactly one arrow $a \to b$. An initial object a is unique up to isomorphism. Indeed, if a' is an initial object, then there are unique arrows $a \to a'$ and $a' \to a$, the unique arrow $a \to a' \to a$ must be $a \xrightarrow{1_a} a$, and similarly for the unique arrow $a' \to a \to a'$. Therefore the arrows $a \to a'$ and $a' \to a$ are inverses of each other.

2. a is **terminal** if for every object b in C, there is exactly one arrow $b \to a$. By an argument similar to the one we made for an initial object, a terminal object is unique up to isomorphism.
3. a is a **null object** if it is both initial and terminal. For example, in the category R-**Mod**, the zero module 0 is a null object. The null object, if it exists, is unique up to isomorphism.

Fix a category C with a null object z, and fix objects a and b of C. Let f be the unique arrow $a \to z$ and g be the unique arrow $z \to b$. The **zero arrow** from a to b, denoted $a \xrightarrow{0} b$, is the composite arrow $g\ f$.

6. Functors

Fix categories $C = (O, A, I, \circ)$ and $C' = (O', A', I', \circ')$. A **functor** F from C to C' is a pair of maps $F\colon O \to O'$ and $F\colon A \to A'$ such that

1. For all objects a in O, $F\ 1_a = 1_{F\ a}$.
2. One of the following conditions holds:
   a. F is **covariant**:
      i. For all arrows $a \xrightarrow{f} b$ in A, there is a corresponding arrow $F\ a \xrightarrow{F\ f} F\ b$ in A'.
      ii. For all pairs of composable arrows (f, g) in $A \times A$, $F(g \circ f) = F\ g \circ F\ f$.
   b. F is **contravariant**:
      i. For all arrows $a \xrightarrow{f} b$ in A, there is a corresponding arrow $F\ b \xrightarrow{F\ f} F\ a$ in A'.
      ii. For all pairs of composable arrows (f, g) in $A \times A$, $F(g \circ f) = F\ f \circ F\ g$.

   Covariant functors preserve the directions of the arrows, and contravariant functors reverse the directions.

In this document, we will write $F\colon C \to C'$ to denote a covariant functor F from C to C'. We will write a contravariant functor F from C to C' as $F\colon C' \leftarrow C$. See § 8 for an alternate notation.

Fix a category C. Let V be a set containing (1) every subset of the set of arrows of C and (2) every mapping between any two such subsets. We associate functors with the hom sets of C as follows:

1. Fix an object b of C. Define the functor $\hom_C(-, b) = C(-, b)\colon \mathbf{Set}_V \leftarrow C$ by the maps $a \mapsto C(a, b)$ and $f \mapsto C(f, 1_b)$. This functor is contravariant because it maps $f\colon a' \to a$ to $h \mapsto h\ f\colon C(a, b) \to C(a', b)$.
2. Fix an object a of C. Define the functor $C(a, -)\colon C \to \mathbf{Set}_V$ by the maps $b \mapsto C(a, b)$ and $g \mapsto C(1_a, g)$. This functor is covariant because it maps $g\colon b \to b'$ to $h \mapsto g\ h\colon C(a, b) \to C(a, b')$.

When C has small hom sets, we can replace $\mathbf{Set}_V$ with **Set**.

An ordered pair of functors $(F\colon C_F \to C'_F, G\colon C_G \to C'_G)$ is **composable** when $C_G = C'_F$. Given a pair of composable functors, define the functor $G \circ F\colon C_F \to C'_G$ by the maps $a \mapsto G(F\ a)$ and $f \mapsto G(F\ f)$. This functor is called the **composition** associated with the pair (F, G). It is associative. As when composing arrows (§ 2), we often write $G\ F$ instead of $G \circ F$.

Fix a category C. The **identity functor** for C, denoted I_C, is the functor defined by the identity maps $a \mapsto a$ and $f \mapsto f$. It is both covariant and contravariant.

Let V be a set containing some categories and all the mappings defining covariant functors between pairs of those categories. By the previous two paragraphs, we can form $\mathbf{Cat}_V$, the category of all categories in V, with covariant functors in V as arrows. If $V = U$ (i.e., if our universe of small or ordinary mathematical objects contains all the categories of interest), then we can form the category **Cat** of all small categories.

Fix categories $C = (O, A, I, \circ)$ and $C' = (O', A', I', \circ')$. A functor $F\colon C \to C'$ is an **isomorphism** if the maps $F\colon O \to O'$ and $F\colon A \to A'$ are bijections. In this case, each map has an inverse, and so there is an inverse functor $F^{-1}\colon C' \to C$ such that $F^{-1}\ F = I_C$ and $F\ F^{-1} = I_{C'}$.

A functor that "forgets" some structure of a category is called a **forgetful functor**. For example, we may define a forgetful functor from **Grp** to **Set** that maps each object (group) in **Grp** to its underlying set and each arrow (group homomorphism) in **Grp** to its underlying map of sets. The functor forgets the group structure associated with the sets and maps.

Fix categories $C = (O, A, I, \circ)$ and $C' = (O', A', I', \circ')$ and a functor $F\colon C \to C'$. For each pair of objects a and b in C, F induces a map $F_{a,b}\colon C(a, b) \to C(F\ a, F\ b)$; it is just the map $F\colon A \to A'$ restricted to $C(a, b) \subseteq A$.

1. F is **full** if, for every pair of objects a and b in C, $F_{a,b}$ is surjective.
2. F is **faithful** if, for every pair of objects a and b in C, $F_{a,b}$ is injective.

A functor that is both full and faithful is called **fully faithful**. In this case, every map $F_{a,b}$ is a bijection. However, this does not imply that F is an isomorphism, because the map $F\colon O \to O'$ may not be a bijection.

Fix a category $C = (O, A, I, \circ)$ and a subcategory $C' = (O', A', I', \circ')$ of C (§ 2). The inclusion maps $S\colon O' \to O$ and $S\colon A' \to A$ form a faithful functor S, called the **inclusion functor**. If S is full, then we say that C' is a **full subcategory** of C. For example, the category of finite sets is a full subcategory of **Set**.

A category C is **concrete** if there exists a faithful functor $F\colon C \to \mathbf{Set}_V$ for some V. In this case, we may identify each object a in C with the set $F\ c$ and each arrow $a \xrightarrow{f} b$ in C with the mapping of sets $F\ f\colon F\ a \to F\ b$. Examples of concrete categories include $\mathbf{Set}_V$, $\mathbf{Grp}_V$, and R-$\mathbf{Mod}_V$.

7. Natural Transformations

Fix categories $C = (O, A, I, \circ)$ and $C' = (O', A', I', \circ')$ and functors $F\colon C \to C'$ and $G\colon C \to C'$. A **transformation of functors** $\tau\colon F \stackrel{\cdot}{\to} G$ is a mapping $\tau\colon O \to A'$ that assigns to every object a in O an arrow $\tau\ a\colon F\ a \to G\ a$ in A'. Notice that for every object a in O, F maps a to $F\ a$, G maps a to $G\ a$, and the arrow $\tau\ a$ relates the two mappings. We often write τ_a instead of $\tau\ a$. We call τ_a the **component** of the transformation τ at a.

It is often useful to think of a transformation τ as a family of arrows $\{\tau_a\}_{a \in O}$ in C' indexed by the objects of C. We also write $\{\tau_a\}_{a \in C}$, using $a \in C$ to denote an object a of C.

We say that a transformation $\tau\colon F \stackrel{\cdot}{\to} G$ is **natural** if, for every arrow $a \xrightarrow{f} b$ in A, the diagram shown in Figure 3 commutes.

$$\begin{tikzcd}
F\ a \arrow[r, "\tau_a"] \arrow[d, "F\ f"'] & G\ a \arrow[d, "G\ f"] \\
F\ b \arrow[r, "\tau_b"'] & G\ b
\end{tikzcd}$$

Figure 3: The commutative diagram for a natural transformation $\tau\colon F \stackrel{\cdot}{\to} G$.

In this case we call τ a **natural transformation** of the functor F to the functor G. A natural transformation is also called a **morphism of functors**. A natural transformation of contravariant functors is similar, but the directions of $F\ f$ and $G\ f$ in the diagram are reversed.

Fix a natural transformation $\tau\colon F \stackrel{\cdot}{\to} G$. If each arrow τ_a is invertible in C' (§ 5), then we say that τ is a **natural equivalence** or **natural isomorphism**, and we write $F \cong G$.

An **equivalence** between categories C and C' is a pair of functors $F\colon C \to C'$ and $G\colon C' \to C$ such that $G \circ F \cong I_C$ and $F \circ G \cong I_{C'}$.

Fix a functor $F\colon C \to C'$. The **identity transformation** $F \stackrel{\cdot}{\to} F$ associated with F is the mapping $a \mapsto 1_{F\ a}$. Notice that we have $F\ a \xrightarrow{1_{F\ a}} F\ a$ as required for a transformation of functors, and the transformation is trivially natural.

Fix categories B and C. The **functor category** C^B is defined as follows:

1. The objects are all functors $F\colon B \to C$.
2. The arrows are all natural transformations $\tau\colon F \stackrel{\cdot}{\to} G$ of functors $F\colon B \to C$ and $G\colon B \to C$.
3. For each functor F, the identity arrow 1_F is the identity transformation associated with F.
4. For each composable pair of arrows $(F \xrightarrow{\sigma} G, G \xrightarrow{\tau} H)$, the composition is written $\tau \cdot \sigma$ instead of $\tau \circ \sigma$ or $\tau\ \sigma$. It is defined as the mapping $a \mapsto \tau_a\ \sigma_a$, where the right-hand expression is a composition of arrows $F\ a \to G\ a$ and $G\ a \to H\ a$ in C.

Notice that in item 4 we have $F\ a \xrightarrow{\tau_a\ \sigma_a} H\ a$, as required for the transformation of functors $\tau \cdot \sigma \colon F \dot{\to} H$. This transformation is natural, and the composition is associative (proof omitted).

Let V be a set containing some categories, all the mappings defining covariant functors between those categories, and all the mappings defining natural transformations between the functors. We can construct another category with natural transformations as arrows, as follows:

1. The objects are all the *categories* (not the functors) in V.
2. The arrows $B \xrightarrow{\tau} C$ are all natural transformations $\tau \colon F \dot{\to} G$ with $F \colon B \to C$ and $G \colon B \to C$.
3. For each category C, the identity arrow 1_C is the identity transformation associated with the identity functor I_C.
4. For each composable pair of arrows $(B \xrightarrow{\sigma} C, C \xrightarrow{\tau} D)$, with $\sigma \colon F_\sigma \dot{\to} G_\sigma$ and $\tau \colon F_\tau \dot{\to} G_\tau$, the composition $B \xrightarrow{\tau \circ \sigma} D$ is the natural transformation $\tau \circ \sigma \colon F_\tau\ F_\sigma \dot{\to} G_\tau\ G_\sigma$ given by the mapping $a \mapsto G_\tau\ \sigma\ a \circ \tau\ F_\sigma\ a$. Here the right-hand expression is the composition of arrows $F_\tau\ F_\sigma\ a \to G_\tau\ F_\sigma\ a$ and $G_\tau\ F_\sigma\ a \to G_\tau\ G_\sigma\ a$ in D.

In item 4, the composition may also be defined as $a \mapsto \tau\ G_\sigma\ a \circ F_\tau\ \sigma\ a$. The two definitions are equivalent, the composition is a natural transformation, and the composition is associative (proof omitted).

The two ways of composing natural transformations satisfy the following **interchange law** (proof omitted):

$$(\tau' \cdot \sigma') \circ (\tau \cdot \sigma) = (\tau' \circ \tau) \cdot (\sigma' \circ \sigma).$$

We may compose natural transformations with functors as follows. Let $\tau \colon F \dot{\to} G$ be a natural transformation of functors $F \colon C \to D$ and $G \colon C \to D$.

1. Fix a category B and a functor $H \colon B \to C$. Write $\tau\ H$ to denote the composite transformation $a \mapsto \tau\ (H\ a)$. Then $(\tau\ H)\ a = \tau\ (H\ a)$ is an arrow from $F\ (H\ a) = (F\ H)\ a$ to $G\ (H\ a) = (G\ H)\ a$, so $\tau\ H$ is a transformation from the composite functor $F\ H$ to the composite functor $G\ H$. It is in fact a natural transformation, because for any arrow $a \xrightarrow{f} b$ in B, we have
$$\begin{aligned} ((G\ H)\ f) \circ ((\tau\ H)\ a) &= (G\ (H\ f)) \circ (\tau\ (H\ a)) \\ &= (\tau\ (H\ b)) \circ (F\ (H\ f)) \quad \text{(naturality of } \tau) \\ &= ((\tau\ H)\ b) \circ ((F\ H)\ f). \end{aligned}$$
Therefore $\tau\ H$ is a natural transformation $\tau\ H \colon F\ H \dot{\to} G\ H$.
2. Fix a category B and a functor $H \colon D \to B$. Write $H\ \tau$ to denote the composite transformation $a \mapsto H\ (\tau\ a)$. Then $(H\ \tau)\ a = H\ (\tau\ a)$ is an arrow from $H\ (F\ a) = (H\ F)\ a$ to $H\ (G\ a) = (H\ G)\ a$, so $H\ \tau$ is a transformation from the composite functor $H\ F$ to the composite functor $H\ G$. It is in fact a natural transformation, because for any arrow $a \xrightarrow{f} b$ in C, we have
$$\begin{aligned} ((H\ G)\ f) \circ ((H\ \tau)\ a) &= (H\ (G\ f)) \circ (H\ (\tau\ a)) \\ &= H\ ((G\ f) \circ (\tau\ a)) \quad \text{(definition of a functor)} \\ &= H\ ((\tau\ b) \circ (F\ f)) \quad \text{(naturality of } \tau) \\ &= (H\ (\tau\ b)) \circ (H\ (F\ f)) \quad \text{(definition of a functor)} \\ &= ((H\ \tau)\ b) \circ ((H\ F)\ f). \end{aligned}$$
Therefore $H\ \tau$ is a natural transformation $H\ \tau \colon H\ F \dot{\to} H\ G$.

This form of composition will be useful when we discuss adjoint functors (§ 14).

8. Duality and Opposite Categories

Duality: For any statement S about a category, the **dual** statement S^* is the statement constructed by transforming S as follows:

1. Replace each arrow $a \xrightarrow{f} b$ with $b \xrightarrow{f} a$.
2. Replace each composition $f\ g$ with $g\ f$.
3. Replace each property P defined by a statement T with the dual statement T^*. If possible, replace T^* with an equivalent property P^*.

As an example of step 3, suppose S refers to the domain of f, where f is the arrow $a \xrightarrow{f} b$. In forming S^*, we replace "domain of f" with its definition, i.e., "the object a in the arrow $a \xrightarrow{f} b$." Then we form the dual of the definition, i.e., "the object a in the arrow $b \xrightarrow{f} a$." Then we see that a is the codomain of f, so we replace "domain of f" in S with "codomain of f" in S^*. Other examples of dual properties P and P^* are the following:

- The property of being a monic or epi arrow.
- The property of being a left or right inverse of an arrow.
- The property of being an initial or terminal object.

It is an easy exercise to verify that in each case, the definitions given in § 5 are dual to each other.

Duality has the property that for any statement S, $S^{**} = S$. We consider a diagram (§ 3) to be a statement about a category; the dual of a diagram has all its arrows and compositions reversed.

Fix a statement S and a sequence of statements $\{S_i\}$ proving that S is true. Then the sequence of statements $\{S_i{}^*\}$ is a proof that S^* is true. This is called the **duality principle** in category theory: whenever we can prove a statement S, we get the proof (and the truth) of the dual statement S^* "for free."

Opposite categories: Fix a category C. The **opposite category** of C, denoted C^{op}, is the unique category such that for every statement S that is true about C, the dual statement S^* is true about C^{op}. In particular, for every arrow $a \xrightarrow{f} b$ in C, there is a corresponding arrow $b \xrightarrow{f} a$ in C^{op}; and for every composition $g\ f$ in C, there is a corresponding composition $f\ g$ in C^{op}. Note that $(C^{\mathrm{op}})^{\mathrm{op}} = C$.

For each category C, we define the **opposite functor** $\mathrm{op}_C\colon C^{\mathrm{op}} \leftarrow C$ by the pair of maps taking every object a in C to the object a in C^{op} and every arrow $a \xrightarrow{f} b$ in C to the arrow $b \xrightarrow{f} a$ in C^{op}. It is a contravariant functor. Note that the composition $\mathrm{op}_{C^{\mathrm{op}}}\ \mathrm{op}_C$ is equal to the identity functor I_C.

Fix categories C and D and a contravariant functor $F\colon D \leftarrow C$.

1. We may write $F = F\ I_C = F\ \mathrm{op}_{C^{\mathrm{op}}}\ \mathrm{op}_C$. Therefore $F = G\ \mathrm{op}_C$, where $G\colon C^{\mathrm{op}} \to D$ is the covariant functor $F\ \mathrm{op}_{C^{\mathrm{op}}}$.
2. We may write $F = I_D\ F = \mathrm{op}_{D^{\mathrm{op}}}\ \mathrm{op}_D\ F$. Therefore $F = \mathrm{op}_{D^{\mathrm{op}}}\ H$, where $H\colon C \to D^{\mathrm{op}}$ is the covariant functor $\mathrm{op}_D\ F$.

Thus we may specify any contravariant functor $F\colon D \leftarrow C$ by giving either a covariant functor $G\colon C^{\mathrm{op}} \to D$ (with the understanding that $F = G\ \mathrm{op}_C$) or a covariant functor $H\colon C \to D^{\mathrm{op}}$ (with the understanding that $F = \mathrm{op}_{D^{\mathrm{op}}}\ H$).

Note that a covariant functor $F\colon C \to D$ maps C to a subcategory of D, and a contravariant functor $F\colon D \leftarrow C$ maps C to a subcategory of D^{op} via the covariant functor $H\colon C \to D^{\mathrm{op}}$.

9. Universal Elements and Arrows

The tensor product: In this section, we will use the tensor product of modules as an example. We begin by restating the definition of the tensor product. For more information on tensor products, see my paper *Definitions for Commutative Algebra*.

Fix a commutative ring R and R-modules A and A'. The tensor product $A \otimes_R A'$ is the unique R-module equipped with a "universal" bilinear map $u\colon A \times A' \to A \otimes_R A'$ such that for any R-module B and any bilinear map $f\colon A \times A' \to B$, there exists a unique module homomorphism $g_f\colon A \otimes_R A' \to B$ with $f = g_f \circ u$. Figure 4 shows the commutative diagram. This diagram exists in a category whose objects are R-modules and Cartesian products of R-modules, and whose arrows are homomorphisms of R-modules and bilinear maps from Cartesian products of R-

$$\begin{tikzcd} A \times A' \arrow[r, "u"] \arrow[dr, "f"'] & A \otimes_R A' \arrow[d, "g_f"] \\ & B \end{tikzcd}$$

Figure 4: The commutative diagram for the tensor product of modules A and A'.

modules to R-modules.

We may restate the definition of the tensor product $A \otimes_R A'$ in terms of a functor. Define the functor $S: R$-**Mod** $\to$ **Set** as follows:

- The object function of S takes each R-module M to the set of bilinear maps from $A \times A'$ to M.
- The arrow function of S takes each module homomorphism $g: M \to M'$ to the mapping of sets $S\ g: S\ M \to S\ M'$ given by $f \mapsto g \circ f$. Notice that $g \circ f$ is a bilinear map from $A \times A'$ to M', as required.

Now we have $u \in S(A \otimes_R A')$, $f \in S\ B$, and $g_f \circ u = (S\ g_f)\ u$. Here $(S\ g_f)\ u$ means "apply the mapping $S\ g_f: S\ (A \otimes A') \to S\ B$ to the element $u \in S(A \otimes_R A')$, yielding the element $g_f \circ u \in S\ B$." Therefore we may define the tensor product $A \otimes_R A$ to be the unique R-module such that, for a fixed bilinear map $u \in S(A \otimes_R A')$, any R-module B, and any bilinear map $f \in S\ B$, there exists a unique module homomorphism $g_f: A \otimes_R A' \to B$ with $f = (S\ g_f)\ u$.

Universal elements: The map u in the definition of the tensor product is an example of a **universal element** for a functor to **Set**. The general definition is as follows. Fix a category C, a functor $S: C \to$ **Set**, and an object a in C. An element $u \in S\ a$ is a universal element of $S\ a$ if, for every object b in C and every element $f \in S\ b$, there exists an arrow $a \xrightarrow{g_f} b$ in C with $f = (S\ g_f)\ u$. See Table 1.

General construction	**Tensor product** $A \otimes_R A'$
A category C	The category R-**Mod**
A functor $S: C \to$ **Set**	The functor $S: R$-**Mod** $\to$ **Set** taking each R-module M to the set of bilinear maps $A \times A' \to M$, and taking each R-module homomorphism $g: M \to M'$ to the function $f \mapsto g\ f$ that takes each bilinear map $f: A \times A' \to M$ to the bilinear map $g \circ f: A \times A' \to M'$
An object a in C	The tensor product $A \otimes_R A'$
A universal element $u \in S\ a$	The bilinear map $u: A \times A' \to A \otimes_R A'$ in the definition of the tensor product
For every object b in C	For every R-module B
For every $f \in S\ b$	For every bilinear map $f: A \times A' \to B$
An arrow $a \xrightarrow{g_f} b$ in C	An R-module homomorphism $g_f: A \otimes_R A \to B$
$f = (S\ g_f)\ u$	$f = g_f \circ u$

Table 1: Universal elements.

Universal elements as arrows: Given any set X and any element $x \in X$, we may represent x as an arrow in the category **Set** as follows. Let $*$ represent any set $\{e\}$ containing one element e. Define $x: * \to X$ to be the map $e \mapsto x$. This map is an arrow $* \xrightarrow{x} X$ in the category **Set**, and there is exactly one such arrow for each element x in X. The arrow "picks out" an element x of X by mapping e to it. In this way, we may represent any element of X as an arrow $* \to X$. Another way to say this is that $\hom(*, X) \cong X$.

We may use this construction to restate the definition of a universal element. Fix a category C, a functor $S: C \to$ **Set**, and an object a in C. Then an arrow $* \xrightarrow{u} S\ a$ in **Set** is a universal element of $S\ a$ if, for every object b

in C and every arrow $* \xrightarrow{f} S\ b$ in **Set**, there exists an arrow $a \xrightarrow{g_f} b$ in C with $f = (S\ g_f) \circ u$. Here $(S\ g_f) \circ u$ means "compose the arrow $* \xrightarrow{u} S\ a$ with the arrow $S\ a \xrightarrow{S\ g_f} S\ b$, yielding an arrow $* \to S\ b$."

Figure 5 shows the diagram. Figure 6 shows the same diagram, specialized to the case of the tensor product. Both diagrams exist in the category **Set**.

$$\begin{tikzcd} * \arrow[r, "u"] \arrow[dr, "f"'] & S\ a \arrow[d, "S(a \xrightarrow{g_f} b)"] \\ & S\ b \end{tikzcd}$$

Figure 5: The commutative diagram for a universal element of $S\ a$, with set membership represented as an arrow.

$$\begin{tikzcd} * \arrow[r, "* \mapsto u\colon A \times A' \to A \otimes_R A'"] \arrow[dr, "* \mapsto f\colon A \times A' \to B"'] & S(A \otimes_R A') \arrow[d, "S(g_f\colon A \otimes_R A' \to B)"] \\ & S\ B \end{tikzcd}$$

Figure 6: Figure 5 specialized to the case of the tensor product.

Universal arrows: Now we can generalize the concept of a universal element, by replacing the category **Set** with any category D. Fix categories C and D, a functor $F\colon C \to D$, an object s in D (the "source object"), and an object a in C. An arrow $s \xrightarrow{u} F\ a$ is a **universal arrow** from s to $F\ a$ if, for every object b in C and every arrow $s \xrightarrow{f} F\ b$ in D, there exists an arrow $a \xrightarrow{g_f} b$ in C with $f = (F\ g_f) \circ u$. Figure 7 shows the diagram, in the category D.

$$\begin{tikzcd} s \arrow[r, "u"] \arrow[dr, "f"'] & F\ a \arrow[d, "F(a \xrightarrow{g_f} b)"] \\ & F\ b \end{tikzcd}$$

Figure 7: The commutative diagram for a universal arrow $s \xrightarrow{u} F\ a$.

The definition of a universal element using arrows is a special case of a universal arrow, with $D = \mathbf{Set}$ and $s = *$. Table 2 summarizes the three constructions discussed above (universal elements, universal elements as arrows, and universal arrows).

The dual construct: The dual construct of a universal arrow from s to $F\ a$ is a universal arrow from $F\ a$ to t. Fix categories C and D, a functor $F\colon C \to D$, an object t in D (the "target object"), and an object a in C. An arrow $F\ a \xrightarrow{u} t$ is a universal arrow from $F\ a$ to t if, for every object b in C and every arrow $F\ b \xrightarrow{f} t$, there exists an arrow $b \xrightarrow{g_f} a$ in C with $f = u \circ (F\ g_f)$. Figure 8 shows the diagram. Table 3 summarizes the dual universal constructs.

Universal Elements	**Universal Elements as Arrows**	**Universal Arrows**
A category C	A category C	Categories C and D
A functor $S\colon C \to \mathbf{Set}$	A functor $S\colon C \to \mathbf{Set}$	A functor $F\colon C \to D$
—	The object * in **Set**	An object s in D
An object a in C	An object a in C	An object a in C
A universal element $u \in S\ a$	A universal arrow $* \xrightarrow{u} S\ a$	A universal arrow $s \xrightarrow{u} F\ a$
For every object b in C	For every object b in C	For every object b in C
For every $f \in S\ b$	For every arrow $* \xrightarrow{f} S\ b$	For every arrow $s \xrightarrow{f} F\ b$
An arrow $a \xrightarrow{g_f} b$ in C	An arrow $a \xrightarrow{g_f} b$ in C	An arrow $a \xrightarrow{g_f} b$ in C
$f = (S\ g_f)\ u$	$f = (S\ g_f) \circ u$	$f = (F\ g_f) \circ u$

Table 2: Universal elements and arrows.

$$\begin{tikzcd}
t & F\ a \arrow[l, "u"'] \\
 & F\ b \arrow[ul, "f"] \arrow[u, "F(b \xrightarrow{g_f} a)"']
\end{tikzcd}$$

Figure 8: The commutative diagram for a universal arrow $F\ a \xrightarrow{u} t$.

Universal Arrows to $F\ a$	**Universal Arrows from** $F\ a$
Categories C and D	Categories C and D
A functor $F\colon C \to D$	A functor $F\colon C \to D$
An object s in D	An object t in D
An object a in C	An object a in C
A universal arrow $s \xrightarrow{u} F\ a$	A universal arrow $F\ a \xrightarrow{u} t$
For every object b in C	For every object b in C
For every arrow $s \xrightarrow{f} F\ b$	For every arrow $F\ b \xrightarrow{f} t$
An arrow $a \xrightarrow{g_f} b$ in C	An arrow $b \xrightarrow{g_f} a$ in C
$f = (F\ g_f) \circ u$	$f = u \circ (F\ g_f)$

Table 3: Dual universal constructs.

Comma categories: A universal arrow is an initial object (§ 5) in a category called a **comma category**. The definition of a comma category is as follows. Let B, C, and D be categories, and let $F\colon B \to D$ and $G\colon C \to D$ be functors. The comma category (F, G) or $(F \downarrow G)$ consists of the following elements:

1. The objects are arrows $F\ b \xrightarrow{f} G\ c$ with b an object of B and c an object of C. This notation means that there is a distinct object for each triple $(b, c, F\ b \xrightarrow{f} G\ c)$.
2. The arrows $(F\ b \xrightarrow{f} G\ c) \xrightarrow{g} (F\ b' \xrightarrow{f'} G\ c')$ are pairs of arrows $g = (b \xrightarrow{g_1} b', c \xrightarrow{g_2} c')$ in $B \times C$ such that the diagram shown in Figure 9 commutes.
3. The identity arrow $1_{F\ b \xrightarrow{f} G\ c}$ is $(1_b, 1_c)$.

$$\begin{tikzcd} F\ b \arrow[r, "f"] \arrow[d, "F\ g_1"'] & G\ c \arrow[d, "G\ g_2"] \\ F\ b' \arrow[r, "f'"'] & G\ c' \end{tikzcd}$$

Figure 9: The commutative diagram for an arrow (g_1, g_2) in the category $(F \downarrow G)$.

4. Composition of arrows is given by $(h_1, h_2) \circ (g_1, g_2) = (h_1 \circ g_1, h_2 \circ g_2)$.

Let B be the category with one object o, and let $s\colon B \to D$ be the functor taking the object o in B to the object s in D. Then Figure 10 shows the commutative diagram in Figure 9 applied to an arrow $(s\ o \xrightarrow{u} G\ c) \xrightarrow{(1_o, g_f)} (s\ o \xrightarrow{f} G\ c')$ in the category $(s \downarrow G)$.

$$\begin{tikzcd} s\ o = s \arrow[r, "u"] \arrow[d, "s\ 1_o = 1_s"'] & G\ c \arrow[d, "G\ g_f"] \\ s\ o = s \arrow[r, "f"'] & G\ c' \end{tikzcd}$$

Figure 10: The commutative diagram for an arrow $(1_o, g_f)$ in the category $(s \downarrow G)$.

Unifying the occurrences of s in Figure 10 and renaming G to F, c to a, and c' to b yields the diagram shown in Figure 7. Therefore, with reference to Figure 7, we have the following:

1. The arrow $s \xrightarrow{u} F\ a$ is an object in the category $(s \downarrow F)$.
2. $s \xrightarrow{u} F\ a$ is a universal arrow if and only if for each object $s \xrightarrow{f} F\ b$ in $(s \downarrow F)$, there exists a unique arrow $(s \xrightarrow{u} F\ a) \xrightarrow{(1_o, g_f)} (s \xrightarrow{f} F\ b)$ in $(s \downarrow F)$, i.e., $s \xrightarrow{u} F\ a$ is an initial object in $(s \downarrow F)$.

Uniqueness of universal arrows: In general, an initial object is unique up to isomorphism (§ 5). Therefore, for any functor $F\colon C \to D$, a universal arrow $s \xrightarrow{u} F\ a$ is unique up to isomorphism in $(s \downarrow F)$. Since an arrow in $(s \downarrow F)$ is a pair of arrows $(1_o, f)$, the object a is also unique up to isomorphism in C.

10. Limits and Colimits

In this section we define limits and colimits in category theory. Limits and colimits are examples of universal arrows (§ 9). We must deal with the following notational inconvenience:

1. Limits in category theory correspond to inverse limits in abstract algebra.
2. Colimits in category theory correspond to direct limits in abstract algebra.

This notation is backwards, but standard. It seems to be motivated by the fact that a product in category theory (§ 11) is a special case of a limit in category theory, and a product in category theory does correspond to a product in set theory or abstract algebra.

10.1. Limits

A limit in category theory generalizes the idea of an inverse limit in algebra.

Inverse limits of modules: We begin by recalling the definition of an inverse limit of modules in commutative algebra. For more information on inverse limits, see my paper *Definitions for Commutative Algebra*.

Fix a nonempty partially ordered set I, a ring R, and a family $S = \{A_i\}_{i \in I}$ of R-modules. S is an **inverse system of modules** over I (inverse system for short) if there exist homomorphisms $f_{ij}\colon A_j \to A_i$ for all $i \le j$ such that

1. For all $i \in I$, f_{ii} is the identity map on A_i.

2. For all $i \le j \le k$, $f_{ik} = f_{ij} \circ f_{jk}$.

Fix an inverse system S. The **inverse limit** of S, written $\varprojlim S$, consists of all elements $\{a_i\}_{i \in I}$ of the direct product $\prod_{i \in I} A_i$ such that for all $i \le j$, $a_i = f_{ij}\ a_j$. The inverse limit is a submodule of the direct product.

For each $i \in I$, let $\upsilon_i : \varprojlim S \to A_i$ be the R-module homomorphism given by $\upsilon_i(\{a_j\}_{j \in I}) = a_i$. Then we have the following:

1. For any i and j in I with $i \le j$, $\upsilon_i = f_{ij} \circ \upsilon_j$. Figure 11 shows the commutative diagram.

$$\begin{tikzcd} & \varprojlim S \arrow[dl, "\upsilon_i"'] \arrow[dr, "\upsilon_j"] & \\ A_i & & A_j \arrow[ll, "f_{ij}"] \end{tikzcd}$$

Figure 11: The commutative diagram for an inverse limit.

2. The inverse limit is universal among R-modules with property (1), in the sense that if B is any R-module equipped with a family of homomorphisms $\tau = \{\tau_i : B \to A_i\}_{i \in I}$ with $\tau_i = f_{ij} \circ \tau_j$ for all $i \le j$, then there is a unique homomorphism $g_\tau : B \to \varprojlim S$ with $\tau_i = \upsilon_i \circ g_\tau$ for all i. This homomorphism is given by

$$g_\tau\ b = \{\tau_i\ b\}_{i \in I}.$$

   That is, $g_\tau\ b$ is the element of the direct product $\prod_{i \in I} A_i$ whose component at each i is $\tau_i\ b$.

Inverse limits as universal arrows: We now express the inverse limit of a family modules as a universal arrow from a functor (§ 9).

First we translate each of the elements in the definition of an inverse limit into category theory. In place of a directed set I, we use a category I with the following properties:

1. The arrows in I provide the partial order relation. That is, there is at most one arrow between any pair of objects i and j in I, and $i \le j$ if and only if there is an arrow $i \to j$.
2. The arrows must provide a valid partial order. The rules for composition of arrows provide reflexivity and transitivity, so we need to check only antisymmetry. (We could also use a **preorder**, which is a partial order without antisymmetry.)

In place of an inverse system of modules, we use a functor $S : I^{\mathrm{op}} \to R$-**Mod**. Each module A_i is the image $S\ i$ of the corresponding object i in I^{op}. For all $i \le j$, each homomorphism $f_{ij} : A_j \to A_i$ is the image $S(j \to i)$ of the corresponding arrow in I^{op}. By the composition rules for functors, these arrows satisfy the definition of an inverse system.

Next we go through the required elements for a universal arrow from a functor (see Table 3).

1. The category C is R-**Mod**.
2. The category D is $R\text{-}\mathbf{Mod}^{I^{\mathrm{op}}}$, the category of functors from I^{op} to R-**Mod**. Recall that an arrow in this category is a natural transformation (§ 7).
3. The functor $F : C \to D$ is the **diagonal functor** Δ defined as follows:
   a. The object function of Δ takes each module B to a functor $\Delta\ B : I^{\mathrm{op}} \to R$-**Mod** defined by $(\Delta\ B)\ i = B$ and $(\Delta\ B)(j \to i) = 1_B$ for all i and j in I^{op}.
   b. The arrow function of Δ takes a homomorphism $f : B \to B'$ to the natural transformation $\delta : \Delta\ B \stackrel{\cdot}{\to} \Delta\ B'$ given by $\delta_i = f$ for all i.
4. The object t in D is the functor $S : I^{\mathrm{op}} \to R$-**Mod** defined above.
5. The object a in C is the inverse limit $L = \varprojlim S$.

6. An arrow $F\ a \xrightarrow{u} t$ is a natural transformation $\upsilon\colon \Delta\ L \xrightarrow{\cdot} S$, i.e., a family of arrows $\{(\Delta\ L)\ i \xrightarrow{\upsilon_i} S\ i\}_{\ i \in I}$ such that the diagram shown in Figure 12 commutes for all $i \le j$. Unifying the instances of $L = \underset{\leftarrow}{\lim}\ S$ yields the commutative diagram shown in Figure 11.

$$\begin{tikzcd}
(\Delta\ L)\ j = L \arrow[r, "\upsilon_j"] \arrow[d, "(\Delta\ L)(j \to i) = 1_L"'] & S\ j = A_j \arrow[d, "S(j \to i) = f_{ij}"] \\
(\Delta\ L)\ i = L \arrow[r, "\upsilon_i"'] & S\ i = A_i
\end{tikzcd}$$

Figure 12: The commutative diagram for a natural transformation $\upsilon\colon \Delta\ L \xrightarrow{\cdot} S$.

7. An object b in C is any R-module B.
8. An arrow $F\ b \xrightarrow{f} t$ is a natural transformation $\tau\colon \Delta\ B \xrightarrow{\cdot} S$, i.e., a family of homomorphisms $\{\tau_i\colon B \to A_i\}_{\ i \in I}$ with $\tau_i = f_{ij} \circ \tau_j$ for all $i \le j$,
9. An arrow $b \xrightarrow{g_f} a$ is a homomorphism $g_\tau\colon B \to L$.
10. The identity $f = u \circ (F\ g_f)$ means $\tau = \upsilon \circ (\Delta\ g_\tau)$. That means $\tau_i = \upsilon_i \circ g_\tau$ for all i.

Items 1 through 10 satisfy the definition of a universal arrow. Item 6 satisfies item 1 in the definition of an inverse limit. Item 10 satisfies item 2 in the definition of an inverse limit. Therefore, items 1 through 10 show that an inverse limit $L = \underset{\leftarrow}{\lim}\ S$ is equivalent to universal arrow $\Delta\ L \xrightarrow{\upsilon} S$.

The general definition of a limit: The expression of an inverse limit as a universal arrow is a special case of a **limit** in category theory. Here is the general definition. Fix categories I and C and a functor $S\colon I \to C$. We may think of I as an index category, and S as a system of objects $S\ i = c_i$ in C. Define the diagonal functor $\Delta\colon C \to C^I$ as follows:

1. $\Delta\ c$ is the functor given by $(\Delta\ c)\ i = c$ and $(\Delta\ c)(i \to j) = 1_c$.
2. $\Delta(c \xrightarrow{f} c')$ is the natural transformation $\delta\colon \Delta\ c \xrightarrow{\cdot} \Delta\ c'$ given by $\delta_i = f$ for all i.

A limit of S is an object $L = \mathrm{Lim}\ S$ in C together with a universal arrow $\Delta\ L \xrightarrow{\upsilon} S$. This arrow is a natural transformation, i.e., a family of arrows $\{((\Delta\ L)\ i = \mathrm{Lim}\ S) \xrightarrow{\upsilon_i} S\ i\}_{\ i \in I}$ such that all diagrams of the form shown in Figure 13 commute for all arrows $S(i \to j)$.

$$\begin{tikzcd}
 & \mathrm{Lim}\ S \arrow[dl, "\upsilon_i"'] \arrow[dr, "\upsilon_j"] & \\
S\ i \arrow[rr] & & S\ j
\end{tikzcd}$$

Figure 13: The commutative diagram for the cone from Lim S.

The object $L = \mathrm{Lim}\ S$ is called the **limit object** of the limit. The limit object, if it exists, is unique up to isomorphism. This is a general property of the object a in a universal arrow (§ 9). The natural transformation $\Delta\ L \xrightarrow{\upsilon} S$ is called a **cone**. The limit object is called the **vertex** of the cone. The functor S is called the **base** of the cone. In the case of a limit, we say that the cone goes from the vertex to the base.

The property of being a universal arrow says that for any cone $c \xrightarrow{\tau} S$ from vertex c in C, as shown in Figure 14, there exists a unique arrow $c \xrightarrow{g_\tau} \mathrm{Lim}\ S$ in C such that the diagram shown in Figure 15 commutes for all i.

Table 4 summarizes the definition of a limit Lim S in terms of a universal arrow from a functor.

$$\begin{tikzcd} & c \arrow[dl, "\tau_i"'] \arrow[dr, "\tau_j"] & \\ S\ i \arrow[rr] & & S\ j \end{tikzcd}$$

Figure 14: The commutative diagram for a cone from c.

$$\begin{tikzcd} c \arrow[rr, "g_\tau"] \arrow[dr, "\tau_i"'] & & \mathrm{Lim}\ S \arrow[dl, "\upsilon_i"] \\ & S\ i & \end{tikzcd}$$

Figure 15: The commutative diagram for the arrow $c \xrightarrow{g_\tau} \mathrm{Lim}\ S$.

A Universal Arrow from a Functor	**The Limit** Lim S
A category C	A category C
A category D	The category C^I, where I is an index category
A functor $F\colon C \to D$	The diagonal functor $\Delta\colon C \to C^I$, where for all c, i, and j, $(\Delta\ c)\ i = c$, $(\Delta\ c)(i \to j) = 1_c$, and $(\Delta(c \xrightarrow{f} c'))\ i = \delta_i = f$
An object t in D	A functor $S\colon I \to C$ providing a family of objects $S\ i = c_i$ in C indexed by I
An object a in C	The limit $L = \mathrm{Lim}\ S$
A universal arrow $F\ a \xrightarrow{u} t$	A natural transformation $\upsilon\colon \Delta\ L \stackrel{\cdot}{\to} S$ providing a cone from the vertex Lim S to the base S
For every object b in C	For every object c in C
For every arrow $F\ b \xrightarrow{f} t$	For every natural transformation $\tau\colon \Delta\ c \stackrel{\cdot}{\to} S$ providing a cone from the vertex c to the to the base S
An arrow $b \xrightarrow{g_f} a$ in C	An arrow $c \xrightarrow{g_\tau} \mathrm{Lim}\ S$
$f = u \circ (F\ g_f)$	$\tau = \upsilon \circ (\Delta\ g_\tau)$, i.e., $\tau_i = \upsilon_i \circ g_\tau$ for all i

Table 4: A limit Lim S as a universal arrow from a functor.

10.2. Colimits

A colimit in category theory is the dual construct of a limit (§ 10.1). It generalizes the idea of a direct limit in algebra.

Direct limits of modules: We recall the definition of a direct limit of modules in commutative algebra. For more information on direct limits, see my paper *Definitions for Commutative Algebra*.

Recall that a **directed set** is a nonempty partially ordered set I in which every pair of elements of I has an upper bound, i.e., for any two elements i and j in I there exists an element k in I with $i \leq k$ and $j \leq k$ according to the partial order on I.

Fix a directed set I, a ring R, and a family $S = \{A_i\}_{i \in I}$ of R-modules. S is a **direct system of modules** over I (direct system for short) if there exist homomorphisms $f_{ij}\colon A_i \to A_j$ for all $i \leq j$ such that

1. For all $i \in I$, f_{ii} is the identity map on B_i.
2. For all $i \leq j \leq k$, $f_{ik} = f_{jk} \circ f_{ij}$.

Fix a direct system S. The **direct limit** of S, written $\varinjlim S$, is the disjoint union $\{(a,i)\colon a \in A_i\}_{i \in I}$ of the modules A_i, subject to the equivalence relation $(a,i) \sim (f_{ij}\ a, j)$ for all $i \le j$ and all $a \in A_i$. In particular, if $a \in A_i$ and $a' \in A_j$, then $(a,i) \sim (a', j)$ if and only if there exists k with $i \le k$ and $j \le k$ such that $f_{ik}\ a = f_{jk}\ a'$. The direct limit is an R-module.

For each $i \in I$, let $\upsilon_i\colon A_i \to \varinjlim S$ be the R-module homomorphism that takes each $a \in A_i$ to the equivalence class $[(a,i)]$ of (a,i) in $\varinjlim S$. Then we have the following:

1. For any i and j in I with $i \le j$, $\upsilon_i = \upsilon_j \circ f_{ij}$. Figure 16 shows the commutative diagram.

$$\begin{tikzcd} & \varinjlim S & \\ A_i \arrow[ur, "\upsilon_i"] \arrow[rr, "f_{ij}"'] & & A_j \arrow[ul, "\upsilon_j"'] \end{tikzcd}$$

Figure 16: The commutative diagram for a direct limit.

2. The direct limit is universal among R-modules with property (1), in the sense that if B is any R-module equipped with a family of homomorphisms $\tau = \{\tau_i\colon A_i \to B\}_{i \in I}$ with $\tau_i = \tau_j \circ f_{ij}$ for all $i \le j$, then there is a unique homomorphism $g_\tau\colon \varinjlim S \to B$ with $\tau_i = g_\tau \circ \upsilon_i$ for all i. This homomorphism is given by

$$g_\tau\ [(a,i)] = \tau_i\ a,$$

where $[(a,i)]$ is the equivalence class in $\varinjlim S$ represented by the element (a,i) in the disjoint union. This homomorphism is well-defined, because if $[(a', j)]$ is any other representative of the same equivalence class, then there exists k such that $f_{ik}\ a = f_{jk}\ a'$, so we have

$$g_\tau\ [(a,i)] = \tau_i\ a = (\tau_k \circ f_{ik})\ a = \tau_k(f_{ik}\ a) = g_\tau\ [(f_{ik}\ a, k)] = g_\tau\ [(f_{jk}\ a', k)]$$

$$= \tau_k(f_{jk}\ a') = (\tau_k \circ f_{jk})\ a' = \tau_j\ a' = g_\tau\ [(a', j)].$$

Direct limits as universal arrows: We now express the direct limit of a family modules as a universal arrow to a functor (§ 9).

In place of a directed set I, we use a category I with the following properties:

1. The arrows of I form a partial order as discussed in § 10.1.
2. The arrows provide an upper bound, i.e., for any objects i and j, there exists an object k with arrows $i \to k$ and $j \to k$.

In place of a direct system of modules, we use a functor $S\colon I \to R$-**Mod**. Each module A_i is the image $S\ i$ of the corresponding object i in I. For all $i \le j$, each homomorphism $f_{ij}\colon A_i \to A_j$ is the image $S(i \to j)$ of the corresponding arrow in I. By the composition rules for functors, these arrows satisfy the definition of a direct system.

Next we go through the required elements for a universal arrow to a functor (see Table 3).

1. The category C is R-**Mod**.
2. The category D is R-$\mathbf{Mod}^I$, the category of functors from I to R-**Mod**. Recall that an arrow in this category is a natural transformation (§ 7).
3. The functor $F\colon C \to D$ is the **diagonal functor** Δ defined as follows:
   a. The object function of Δ takes each module B to a functor $\Delta\ B\colon I \to R$-**Mod** defined by $(\Delta\ B)\ i = B$ and $(\Delta\ B)(i \to j) = 1_B$ for all i and j in I.
   b. The arrow function of Δ takes a homomorphism $f\colon B \to B'$ to the natural transformation $\delta\colon \Delta\ B \stackrel{\cdot}{\to} \Delta\ B'$ given by $\delta_i = f$ for all i.
4. The object s in D is the functor $S\colon I \to R$-**Mod** defined above.

5. The object a in C is the direct limit $L = \varinjlim S$.
6. An arrow $s \xrightarrow{u} F\ a$ is a natural transformation $\upsilon \colon S \stackrel{\cdot}{\to} \Delta\ L$, i.e., a family of arrows $\{S\ i \xrightarrow{\upsilon_i} (\Delta\ L)\}_{\ i \in I}$ such that the diagram shown in Figure 17 commutes for all $i \le j$. Unifying the instances of $L = \varinjlim S$ yields the commutative diagram shown in Figure 16.

$$\begin{tikzcd}
S\ i = A_i \arrow[r, "\upsilon_i"] \arrow[d, "S(i \to j) = f_{ij}"'] & (\Delta\ L)\ i = L \arrow[d, "(\Delta\ L)(i \to j) = 1_L"] \\
S\ j = A_j \arrow[r, "\upsilon_j"'] & (\Delta\ L)\ j = L
\end{tikzcd}$$

Figure 17: The commutative diagram for a natural transformation $\upsilon \colon S \stackrel{\cdot}{\to} \Delta\ L$.

7. An object b in C is any R-module B.
8. An arrow $s \xrightarrow{f} F\ b$ is a natural transformation $\tau \colon S \stackrel{\cdot}{\to} \Delta\ B$, i.e., a family of homomorphisms $\{\tau_i \colon A_i \to B\}_{\ i \in I}$ with $\tau_i = \tau_j \circ f_{ij}$ for all $i \le j$,
9. An arrow $a \xrightarrow{g_f} b$ is a homomorphism $g_\tau \colon L \to B$.
10. The identity $f = (F\ g_f) \circ \upsilon$ means $\tau = (\Delta\ g_\tau) \circ \upsilon$. That means $\tau_i = g_\tau \circ \upsilon_i$ for all i.

Items 1 through 10 satisfy the definition of a universal arrow. Item 6 satisfies item 1 in the definition of a direct limit. Item 10 satisfies item 2 in the definition of a direct limit. Therefore, items 1 through 10 show that a direct limit $L = \varinjlim S$ is equivalent to universal arrow $S \xrightarrow{\upsilon} \Delta\ L$.

The general definition of a colimit: The expression of a direct limit as a universal arrow is a special case of a **colimit** in category theory. Here is the general definition. Fix categories I and C and a functor $S \colon I \to C$. We may think of I as an index category, and S as a system of objects $S\ i = c_i$ in C. Define the diagonal functor $\Delta \colon C \to C^I$ as in the definition of a limit (§ 10.1).

A colimit of S is an object $L = \text{Colim}\ S$ in C together with a universal arrow $S \xrightarrow{\upsilon} \Delta\ L$. This arrow is a natural transformation, i.e., a family of arrows $\{S\ i \xrightarrow{\upsilon_i} ((\Delta\ L)\ i = \text{Colim}\ S)\}_{\ i \in I}$ such that all diagrams of the form shown in Figure 18 commute for all arrows $S(i \to j)$.

$$\begin{tikzcd}
& \text{Colim}\ S & \\
S\ i \arrow[ur, "\upsilon_i"] \arrow[rr] & & S\ j \arrow[ul, "\upsilon_j"']
\end{tikzcd}$$

Figure 18: The commutative diagram for the cone to Colim S.

The object $L = \text{Colim}\ S$ is called the **colimit object** of the colimit. The colimit object, if it exists, is unique up to isomorphism. This is a general property of the object a in a universal arrow (§ 9). As in the case of a limit (§ 10.1), the natural transformation $S \xrightarrow{\upsilon} \Delta\ L$ is called a **cone**. The colimit object is called the **vertex** of the cone. The functor S is called the **base** of the cone. In the case of a colimit, we say that the cone goes from the base to the vertex.

The property of being a universal arrow says that for any cone $S \xrightarrow{\tau} c$ to vertex c in C, as shown in Figure 19, there exists a unique arrow Colim $S \xrightarrow{g_\tau} c$ in C such that the diagram shown in Figure 20 commutes for all i.

Table 5 summarizes the definition of a colimit Colim S in terms of a universal arrow to a functor.

$$\begin{tikzcd} & c & \\ S\ i \arrow[ur, "\tau_i"] \arrow[rr] & & S\ j \arrow[ul, "\tau_j"'] \end{tikzcd}$$

Figure 19: The commutative diagram for a cone to c.

$$\begin{tikzcd} c & & \mathrm{Colim}\ S \arrow[ll, "g_\tau"'] \\ & S\ i \arrow[ul, "\tau_i"] \arrow[ur, "\upsilon_i"'] & \end{tikzcd}$$

Figure 20: The commutative diagram for the arrow Colim $S \xrightarrow{g_\tau} c$.

A Universal Arrow to a Functor	**The Colimit** Colim S
A category C	A category C
A category D	The category C^I, where I is an index category
A functor $F\colon C \to D$	The diagonal functor $\Delta\colon C \to C^I$, where for all c, i, and j, $(\Delta\ c)\ i = c$, $(\Delta\ c)(i \to j) = 1_c$, and $(\Delta(c \xrightarrow{f} c'))\ i = \delta_i = f$
An object s in D	A functor $S\colon I \to C$ providing a family of objects $S\ i = c_i$ in C indexed by I
An object a in C	The colimit $L = $ Colim S
A universal arrow $s \xrightarrow{u} F\ a$	A natural transformation $\upsilon\colon S \stackrel{\cdot}{\to} \Delta\ L$ providing a cone from the base S to the vertex Colim S
For every object b in C	For every object c in C
For every arrow $s \xrightarrow{f} F\ b$	For every natural transformation $S \stackrel{\cdot}{\to} \tau\colon \Delta$ providing a cone from the base S to the vertex c
An arrow $a \xrightarrow{g_f} b$ in C	An arrow Colim $S \xrightarrow{g_\tau} c$
$f = (F\ g_f) \circ u$	$\tau = (\Delta\ g_\tau) \circ \upsilon$, i.e., $\tau_i = g_\tau \circ \upsilon_i$ for all i

Table 5: A colimit Colim S as a universal arrow to a functor.

10.3. Existence of Limits and Colimits

Given categories I and C and a functor $S\colon I \to C$, the limit Lim S may or may not exist, and similarly for the colimit Colim S. When a limit exists, we may demonstrate its existence by explicit construction, as we have done in the preceding sections.

We say a limit or colimit is **small** if the category I is small. A category C is **complete** if all small limits exist in C, **cocomplete** if all small colimits exist in C, and **bicomplete** if it is complete and cocomplete. The categories **Set** and R-**Mod** are bicomplete.

11. Products and Coproducts

In this section we define products and coproducts in category theory. Products and coproducts are important special cases of limits (§ 10.1) and colimits (§ 10.2), respectively. As with limits and colimits, a given product or coproduct may or may not exist in a given category. If the product or coproduct exists, we may demonstrate its existence by explicit construction.

11.1. Products

A product in category theory generalizes the idea of a product in set theory or algebra. A product is a limit Lim S (§ 10.1) in which the index category I is a discrete category (§ 2). In other words, the only arrows in I are the identity arrows 1_i. We denote the product $\prod_{i \in I} S\ i$ or $\prod_{i \in I} a_i$, where $S\ i = a_i$ for all i. We now give some examples of products.

The Cartesian product of sets: Let I be a set, and let $S = \{A_i\}_{i \in I}$ be a family of sets indexed by I. The Cartesian product of S, written $\times_{i \in I} A_i$, is the set of all families of elements $\{a_i\}_{i \in I}$ with $a_i \in A_i$ for all i. When $I = \{1, \ldots, n\}$, we may represent each element of the Cartesian product as an ordered tuple of elements. For example, in the case $I = \{1, 2\}$, the Cartesian product $A_1 \times A_2$ is the set of all ordered pairs (a_1, a_2) with $a_1 \in A_1$ and $a_2 \in A_2$.

Let $P = \times_{i \in I} A_i$ be the Cartesian product of S. For each $i \in I$, let $\upsilon_i : P \to A_i$ be the map given by $\upsilon_i(\{a_j\}_{j \in I}) = a_i$. υ_i is called the ith **projection map**, because it projects an element of the Cartesian product onto A_i. The family of maps $\upsilon = \{\upsilon_i : P \to A_i\}_{i \in I}$ is universal, in the sense that if B is any set equipped with a family of maps $\tau = \{\tau_i : B \to A_i\}_{i \in I}$, then there is a unique map $g_\tau : B \to P$ with $\tau_i = \upsilon_i \circ g_\tau$ for all i. This map is given by

$$g_\tau\ b = \{\tau_i\ b\}_{i \in I}.$$

That is, $g_\tau\ b$ is the element of the Cartesian product $\times_{i \in I} A_i$ whose component at each i is $\tau_i\ b$.

The Cartesian product is a product in the category **Set**. Let I be a discrete category, and fix a functor $S : I \to$ **Set**. Then S represents a family of sets $\{A_i\}_{i \in I}$, where $A_i = S\ i$. It is evident from the definitions that $\times_{i \in I} A_i = \varprojlim S$. Indeed, there are no arrows $i \to j$ in I when $i \neq j$, so there are no diagrams of the form shown in Figures 13 and 14. There are just maps υ_i and τ_i satisfying the diagram shown in Figure 15. This diagram establishes the universal property of the Cartesian product stated above. Since the object satisfying the universal property is unique up to isomorphism, the limit is the Cartesian product.

The direct product of modules: Fix a commutative ring R, let I be a set, and let $S = \{A_i\}_{i \in I}$ be a family of R-modules indexed by I. The direct product of S, written $\prod_{i \in I} A_i$, is the Cartesian product $\times_{i \in I} A_i$ with the R-module structure given by

$$\{a_i\}_{i \in I} + \{a_i'\}_{i \in I} = \{a_i + a_i'\}_{i \in I}$$

and

$$r\ \{a_i\}_{i \in I} = \{r a_i\}_{i \in I}.$$

The direct product is a limit in the category R-**Mod**. Indeed, the direct product is just the inverse limit (§ 10.1) in the case when the only maps f_{ij} are the identity maps f_{ii}, and we have already shown that the inverse limit is a limit.

The product of categories: We may form the product of a family of categories in **Cat**. Let I be a discrete category, let $S : I \to$ **Cat** be a functor, and let $A_i = S\ i$ for all i. We construct the product category $P = \prod_{i \in I} A_i$ as follows:

1. The objects of P are families $a = \{a_i\}_{i \in I}$, with a_i an object of A_i for all i.
2. The arrows $\{a_i\}_{i \in I} \xrightarrow{f} \{a_i'\}_{i \in I}$ of P are families $f = \{a_i \xrightarrow{f_i} a_i'\}_{i \in I}$, with each f_i an arrow in A_i.
3. Composition of arrows in P is given by $\{a_i' \xrightarrow{g_i} a_i''\}_{i \in I} \circ \{\ a_i \xrightarrow{f_i} a_i'\ \}_{i \in I} = \{\ a_i \xrightarrow{g_i \circ f_i} a_i''\}_{i \in I}$.
4. The arrows $\upsilon_i : P \to A_i$ are functors defined by $\upsilon_i(\{a_j\}_{j \in I}) = a_i$ and $\upsilon_i(\{f_j\}_{j \in I}) = f_i$.
5. For any category B and family of arrows $\{\tau_i : B \to A_i\}_{i \in I}$, the arrow $g_\tau : B \to P$ is the functor defined by $g_\tau\ b = \{\tau_i\ b\}_{i \in I}$ and $g_\tau\ f = \{\tau_i\ f\}_{i \in I}$.

It is straightforward to check that this definition satisfies the requirements for a product.

We often write the product of two categories A and B as $A \times B$. Similarly, we write the elements and arrows as ordered pairs (a, b) and (f, g). In this notation, we have $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$, $\upsilon_1(f, g) = f$, and $\upsilon_2(f, g) = g$.

A functor $F\colon A \times B \to C$ from a product of two categories is called a **bifunctor**. We may think of a bifunctor $F\colon A \times B \to C$ as a functor $F(-,-)$ with two arguments, one from A and one from B. Each object b in B yields a functor $F(-,b)\colon A \to C$. The object function of this functor is $a \mapsto F(a,b)$, and the arrow function is $f \mapsto F(f, 1_b)$. Similarly, each object a in A yields a functor $F(a,-)\colon B \to C$.

An example is the functor $\hom_C(-,-) = C(-,-)\colon C^{\mathrm{op}} \times C \to \mathbf{Set}_V$ defined by $(a,b) \mapsto C(a,b)$ and $(f,g) \mapsto C(f,g)$. We have already discussed the functors $C(-,b)$ and $C(a,-)$ in § 6.

Fix categories A, B, and C and functors $F\colon A \to C$ and $B \to C$. The product of the functors F and G, written $F \times G$, is the bifunctor from $A \times B$ to C given by $(F \times G)\ (a,b) = (F\ a, G\ b)$ and $(F \times G)\ (f,g) = (F\ f, G\ g)$.

Fix bifunctors $F\colon A \times B \to C$ and $G\colon A \times B \to C$ and a transformation τ, not necessarily natural, from F to G. That means τ is a family of arrows $\{F(a,b) \xrightarrow{\tau_{(a,b)}} G(a,b)\}_{(a,b)\,\in\, A \times B}$, not necessarily satisfying the commutative diagram shown in Figure 3. For each object b in B, τ yields a transformation $\tau_{(-,b)}\colon F(-,b) \to G(-,b)$ whose component at each a in A is $\tau_{(a,b)}$. Similarly, for each object a in A, τ yields a transformation $\tau_{(a,-)}\colon F(a,-) \to G(a,-)$ whose component at each b in B is $\tau_{(a,b)}$. If $\tau_{(-,b)}$ is a natural transformation for all objects b in B, then we say that the transformation τ is **natural in** a. Similarly, if $\tau_{(a,-)}$ is a natural transformation for all objects a in A, then we say that τ is **natural in** b. τ is a natural transformation of bifunctors if and only if it is natural in a and b (proof omitted).

11.2. Coproducts

The coproduct is the dual construction to the product. It is a colimit (§ 10.2) in which the index category I is a discrete category. We denote the coproduct $\bigoplus_{i \in I} S\ i$ or $\bigoplus_{i \in I} A_i$, where $S\ i = A_i$ for all i. We now give some examples of coproducts.

The disjoint union of sets: Let I be a set, and let $S = \{A_i\}_{i \in I}$ be a family of sets indexed by I. The disjoint union of S, written $\biguplus_{i \in I} A_i$, is the set $\bigcup_{i \in I} (A_i \times \{i\})$ consisting of all pairs (a,i) such that $a \in A_i$. The union is "disjoint" in the sense that when the sets A_i share elements, each A_i contributes a separate copy of each of its elements. For example, let $I = \{1,2\}$, $A_1 = \mathbf{Z}$, and $A_2 = \mathbf{Z}$. Then the union $\mathbf{Z} \cup \mathbf{Z}$ contains one copy of the element 3, but the disjoint union $\mathbf{Z} \uplus \mathbf{Z}$ contains two copies $(3,1)$ and $(3,2)$.

Let $U = \biguplus_{i \in I} A_i$ be the disjoint union of S. For each $i \in I$, let $\upsilon_i\colon A_i \to U$ be the map given by $\upsilon_i\ a = (a,i)$. υ_i is called the ith **injection map**, because it injects an element of A_i into the disjoint union. The family of maps $\upsilon = \{\upsilon_i\colon A_i \to U\}_{i \in I}$ is universal, in the sense that if B is any set equipped with a family of maps $\tau = \{\tau_i\colon A_i \to B\}_{i \in I}$, then there is a unique map $g_\tau\colon U \to B$ with $\tau_i = g_\tau \circ \upsilon_i$ for all i. This map is given by

$$g_\tau\ (a,i) = \tau_i\ a.$$

The disjoint union is a coproduct in the category **Set**. Let I be a discrete category, and fix a functor $S\colon I \to \mathbf{Set}$. By an argument similar to the one that we made for the Cartesian product (§ 11.1), the disjoint union $\biguplus_{i \in I} A_i$ is the colimit Colim S in **Set**.

The direct sum of modules: Fix a commutative ring R, let I be a set, and let $S = \{A_i\}_{i \in I}$ be a family of R-modules indexed by I. The direct sum of S, written $\bigoplus_{i \in I} A_i$, is the submodule of the direct product $\prod_{i \in I} A_i$ consisting of all families $\{a_i\}_{i \in I}$ such that all but finitely many of the elements a_i are zero. Equivalently, $\bigoplus_{i \in I} A_i$ is the R-module generated by the disjoint union $\biguplus_{i \in I} A_i$, i.e., the set of all formal finite sums $\sum_{j=1}^{n} (a_j, i_j)$ with each $a_j \in A_{i_j}$, modulo the relation that two sums s_1 and s_2 are equivalent if each can be put into a common form s_3 by reordering terms and transforming pairs of terms $(a,i) + (a',i)$ into $(a + a', i)$. Note that if I is finite, then $\bigoplus_{i \in I} A_i = \prod_{i \in I} A_i$.

The direct sum $D = \bigoplus_{i \in I} A_i$ is a colimit in the category R-**Mod**. For each i, let $\upsilon_i\colon A_i \to D$ be the R-module homomorphism that takes $a \in A_i$ to the element $\{a_j\}_{j \in I}$ in the direct sum such that $a_i = a$ and $a_j = 0$ for $j \neq i$. Equivalently, $\upsilon_i\ a$ is the formal finite sum consisting of the single term (a,i). For any module B and family of R-module homomorphisms $\{\tau_i\colon A_i \to B\}_{i \in I}$, let map $g_\tau\colon D \to B$ be the R-module homomorphism given by

$$g_\tau \{a_i\}_{i \in I} = \sum_{i \in I} \tau_i\ a_i.$$

This map is well-defined, because all but finitely many of the elements a_i are zero, so the sum on the right-hand side has finitely many nonzero terms. Equivalently, using the definition of the direct sum as a formal finite sum, we can write

$$g_\tau \sum_{j=1}^{n}(a_j, i_j) = \sum_{j=1}^{n} \tau_{i_j}\ a_j.$$

It is a straightforward exercise to show that, with these homomorphisms, the direct sum $\underset{i \in I}{\oplus} A_i$ is the colimit Colim S in the category R-**Mod**, where $S\ i = A_i$. Note that the direct sum is not a direct limit, because the discrete category I is not a directed set (it is not true that every pair of elements in I has an upper bound).

The coproduct of categories: We may form the coproduct of a family of categories in **Cat**. Let I be a discrete category, let $S\colon I \to$ **Cat** be a functor, and let $A_i = S\ i$ for all i. We construct the coproduct category $C = \underset{i \in I}{\oplus} A_i$ as follows:

1. The objects of C are all pairs (a, i), where $i \in I$ and a is an object of A_i.
2. The arrows $(a, i) \to (b, i)$ of C are all pairs $(a \xrightarrow{f} b, i)$, where $i \in I$ and $a \xrightarrow{f} b$ is an arrow of A_i. There are no arrows $(a, i) \to (b, j)$ when $i \neq j$.
3. The identity arrow $1_{(a,i)}$ is $(1_a, i)$.
4. Composition of arrows in C is given by $\{\ (b \xrightarrow{g} c, i)\ \} \circ \{\ (a \xrightarrow{f} b, i)\ \} = \{\ (a \xrightarrow{g \circ f} c, i)\ \}$.
5. The arrows $\upsilon_i\colon A_i \to C$ are functors defined by $\upsilon_i\ a = (a, i)$ and $\upsilon_i\ f = (f, i)$.
6. For any category B and family of arrows $\{\tau_i\colon A_i \to B\}_{i \in I}$, the arrow $g_\tau\colon C \to B$ is the functor defined by $g_\tau(a, i) = \tau_i\ a$ and $g_\tau(f, i) = \tau_i\ f$.

It is straightforward to check that this definition satisfies the requirements for a coproduct.

12. More Examples of Universal Arrows and Elements

Having discussed limits and colimits (§ 10) and products and coproducts (§ 11), we now discuss some more examples of universal arrows and elements (§ 9).

Free categories: A **graph** is a partially specified category: it has objects and arrows, but no identity arrows or composition of arrows. More precisely, a graph is a pair $G = (O, A)$, where O is a set of objects and A is a set of arrows $a \xrightarrow{f} b$, with a and b in O. A graph is also called a **diagram scheme**.

A **morphism** of graphs is a covariant functor (§ 6), minus the requirements about mapping identity arrows and composite arrows. More precisely, if $G = (O, A)$ and $G' = (O', A')$, then a morphism $M\colon G \to G'$ is a pair of maps $M\colon O \to O'$ and $M\colon A \to A'$ such that for all arrows $a \xrightarrow{f} b$ in A, there is a corresponding arrow $M\ a \xrightarrow{M\ f} M\ b$ in A'.

Let V be a set. $\mathbf{Grph}_V$ is the category whose objects are all graphs $G = (O, A)$ with O and A in V and whose arrows are all morphisms between such graphs. When $V = U$, we write **Grph** to denote the category of all small graphs.

Every category $C = (O, A, I, \circ)$ has an underlying graph $G_C = (O, A)$. Fix categories $C = (O, A, I, \circ)$ and $C' = (O', A', I', \circ')$. Every functor $F\colon C \to C'$ is also a morphism $M_F\colon G_C \to G_{C'}$. The forgetful functor $U\colon \mathbf{Cat}_V \to \mathbf{Grph}_V$ takes each category C to $U\ C = G_C$ and each functor F to $U\ F = M_F$.

Fix a graph $G = (O, A)$. The **free category** generated by G is the category $C_G = (O', A', I, \circ)$ defined as follows:

1. $O' = O$.
2. A' is the set of chains $a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n$ of zero or more arrows in A. Every such chain of arrows in A is an arrow $a_0 \to a_n$ in A'. In particular, every arrow in A is a chain $a \xrightarrow{f} b$, so A is included in A'.
3. For each object a in O, 1_a is the chain of zero arrows with $a_0 = a$.
4. Composition works by concatenating composable chains, i.e., the composition of $a \xrightarrow{f} b$ and $b \xrightarrow{g} c$ is $g \circ f = a \xrightarrow{f} b \xrightarrow{g} c$.

If G is an object of $\mathbf{Grph}_V$, then C_G is an object of $\mathbf{Cat}_V$.

Fix a graph $G = (O, A)$ in $\mathbf{Grph}_V$, and let $C_G = (O, A', I, \circ)$ be the free category generated by G. Then $U\ C_G = (O, A')$, where U is the forgetful functor described above. Let $u\colon G \to U\ C_G$ be the morphism of graphs given by the identity map $O \to O$ and the inclusion map $A \to A'$. Then u is a universal arrow: for any category B and morphism $f\colon G \to U\ B$, there exists a unique functor $g_f\colon U\ C_G \to U\ B$ such that $f = U\ g_f \circ u$ (proof omitted).

Quotient categories: Fix a category $C = (O, A, I, \circ)$. A **congruence** on C is an equivalence relation $\sim$ on A such that for every pair of arrows (f, f') with $f \sim f'$, and for every pair of arrows (g, h) such that that the composition $h\ f\ g$ is defined, $h\ f'\ g$ is defined and $h\ f\ g \sim h\ f'\ g$. In particular, if $f \sim f'$, then f and f' have the same domain and the same codomain, because we must have $a \xrightarrow{1_a} a \xrightarrow{f} b \xrightarrow{1_b} b \sim a \xrightarrow{1_a} a \xrightarrow{f'} b \xrightarrow{1_b} b$.

Fix a congruence $\sim$ on C. For each $f \in A$, let $[f] \in A/\!\sim$ denote the equivalence class of f modulo $\sim$. The **quotient category** $C/\!\sim$ is $(O, A/\!\sim, I', \circ')$, where I' is the map $a \mapsto [1_a]$, and $\circ'$ is defined by $[g] \circ [f] = [g \circ f]$. The definition of a congruence makes the composition $\circ'$ well-defined.

Let $Q\colon C \to C/\!\sim$ be the functor defined by the maps $a \mapsto a$ and $f \mapsto [f]$. For any functor $F\colon C \to B$, we say that F **respects** the congruence $\sim$ if $F\ f = F\ f'$ whenever $f \sim f'$. For any such functor F, there exists a unique functor $G_F\colon C/\!\sim\ \to B$ such that $F = G_F \circ Q$. In other words, Q is a universal element for the set of functors from C to $C/\!\sim$ that respect $\sim$ (proof omitted).

Powers and copowers: Consider a product $\prod_{i \in I} a_i$ in a category C (§ 11.1). A cone from a vertex c to the base $\{a_i\}_{i \in I}$ is a family of arrows $\{c \xrightarrow{\tau_i} a_i\}_{i \in I}$. (There are no cross arrows, because I is discrete.) The map $\{\tau_i\}_{i \in I} \mapsto g_\tau$ is a bijection $\prod_{i \in I} \hom_C(c, a_i) \cong \hom_C(c, \prod_{i \in I} a_i)$. We may interpret each side of this bijection as a bifunctor from $C \times \prod_{i \in I} a_i$ to $\mathbf{Set}_V$. Then the bijection is a transformation of these functors, and the transformation is natural in c (proof omitted). When the factors in a product are all equal ($a_i = a$ for all i), the product $\prod_{i \in I} a_i = \prod_{i \in I} a$ is called a **power** and written a^I. With this notation, the bijection becomes $\hom_C(c, a)^I \cong \hom_C(c, a^I)$. The power $\hom_C(c, a)^I$ in the category $\mathbf{Set}_V$ represents the set of all functions from I to $\hom_C$.

The dual concept of a power is called a **copower**. Consider a coproduct $\bigoplus_{i \in I} a_i$ in a category C (§ 11.2). A cone from the base $\{a_i\}_{i \in I}$ to a vertex c is a family of arrows $\{a_i \xrightarrow{\tau_i} c\}_{i \in I}$. The map $g_\tau \mapsto \{\tau_i\}_{i \in I}$ is a bijection $\hom_C(\bigoplus_{i \in I} a_i, c) \cong \prod_{i \in I} \hom_C(a_i, c)$. This bijection is a transformation of functors from $\prod_{i \in I} a_i \times C$ to $\mathbf{Set}_V$, natural in c (proof omitted). When the factors in a coproduct are all equal ($a_i = a$ for all i), the coproduct $\bigoplus_{i \in I} a_i = \bigoplus_{i \in I} a$ is called a copower and written $I \cdot a$, so the bijection becomes $\hom_C(I \cdot a, c) \cong \hom_C(a, c)^I$.

Kernels and cokernels: Kernels and cokernels in category theory generalize the corresponding concepts in algebra. First we review the algebraic concepts. Fix a ring R, R-modules A and B, and an R-module homomorphism $f\colon A \to B$.

In the algebra of modules, the **kernel** of f, written $\ker f$, is the R-module $f^{-1}(0)$, i.e., the R-module consisting of all elements $a \in A$ such that $f(a) = 0$. The **image** of f, written $\operatorname{im} f$, is the R-module $f(A)$, i.e., the module of all elements $f(a)$ with $a \in A$. $\operatorname{im} f$ is isomorphic to $A/\ker f$, i.e., A modulo the relation $a \sim a'$ if and only if $a - a' \in \ker f$. We will write an element of $A/\ker f$ as $a \pmod{\ker f}$, for $a \in A$. The kernel has the following universal property:

1. Let $\upsilon\colon \ker f \to A$ be the inclusion map. Then $f\ \upsilon\colon \ker f \to A = 0$, where we write 0 for the zero map $a \mapsto 0$.
2. If C is any R-module and $g\colon C \to A$ is any homomorphism such that $f\ g = 0$, then $\operatorname{im} g \subseteq \ker f$. Let $h_g\colon C \to \ker f$ be $i\ g$, where $i\colon \operatorname{im} g \to \ker f$ is the inclusion map.
3. h_g is the unique map such that $\upsilon\ h_g = g$.

The **cokernel** of f, written $\operatorname{coker} f$, is the R-module $B/\operatorname{im} A$, i.e., B modulo the relation $b \sim b'$ if and only if $b - b' \in \operatorname{im} A$. The cokernel has the following universal property:

1. Let $\upsilon\colon B \to \operatorname{coker} f$ be the projection map $b \mapsto b \pmod{\operatorname{im} f}$. Then $\upsilon\ f\colon A \to \operatorname{coker} f = 0$.
2. If C is any R-module and $g\colon B \to C$ is any homomorphism such that $g\ f = 0$, then $\operatorname{im} f \subseteq \ker g$. Therefore the projection map $q\colon \operatorname{coker} f = B/\ker f \to B/\ker g \cong \operatorname{im} g$ given by

$$b \ (\mathrm{mod\ im}\ f) \mapsto b \ (\mathrm{mod\ ker}\ g),$$

is well-defined. Let $h_g\colon \mathrm{coker}\ f \to C$ be $i\ q$, where i is the inclusion map from $B/\mathrm{ker}\ g \cong \mathrm{im}\ g$ to C.

3. h_g is the unique map such that $h_g\ \upsilon = g$.

Now we translate these concepts to category theory. Fix a category C that contains a null object (§ 5), so that for every pair of objects (a, b) there is a null arrow $a \xrightarrow{0} b$. Fix an arrow $a \xrightarrow{f} b$ in C.

The **kernel** of f (in the category-theoretic sense) is a limit (§ 10.1) whose limit cone has the form shown in Figure 21. The commutative diagram says $f\ \upsilon = 0$. The universal property of the limit says that for any object c and any cone like Figure 21 with vertex c instead of vertex $\mathrm{ker}\ f$ and with arrow $c \xrightarrow{g} a$ instead of $\mathrm{ker}\ f \xrightarrow{\upsilon} a$, there exists a unique arrow $c \xrightarrow{h_g} \mathrm{ker}\ f$ such that $\upsilon\ h_g = g$. This is exactly the universal property that we stated above for the algebraic kernel.

$$\begin{tikzcd}
& \mathrm{ker}\ f = \mathrm{Lim}\ S \arrow[dl, "\upsilon"'] \arrow[dr, "0"] & \\
a \arrow[rr, "f"'] & & b
\end{tikzcd}$$

Figure 21: The limit cone for $\mathrm{ker}\ f$.

The **cokernel** of f (again in the category theoretic sense) is the dual of the kernel. It is a colimit (§ 10.2) whose colimit cone has the form shown in Figure 22. The commutative diagram says $\upsilon\ f = 0$. The universal property of the colimit says that for any object c and any cone like Figure 22 with vertex c instead of vertex $\mathrm{coker}\ f$ and with arrow $b \xrightarrow{g} c$ instead of $b \xrightarrow{\upsilon} \mathrm{coker}\ f$, there exists a unique arrow $\mathrm{coker}\ f \xrightarrow{h_g} c$ such that $h_g\ \upsilon = g$. This is exactly the universal property that we stated above for the algebraic cokernel.

$$\begin{tikzcd}
& \mathrm{coker}\ f = \mathrm{Colim}\ S & \\
a \arrow[ur, "0"] \arrow[rr, "f"'] & & b \arrow[ul, "\upsilon"']
\end{tikzcd}$$

Figure 22: The colimit cone for $\mathrm{coker}\ f$.

Equalizers and coequalizers: Equalizers and coequalizers generalize kernels and cokernels. They are available in categories that do not have zero objects and zero arrows. Fix a category C and arrows $a \xrightarrow{f_1} b$ and $a \xrightarrow{f_2} b$ in C.

The **equalizer** of f_1 and f_2, which we will denote $\mathrm{eq}(f_1, f_2)$, is a limit whose limit cone has the form shown in Figure 23. The arrow υ "equalizes" the arrows f_1 and f_2, in the sense that $f_1\ \upsilon = f_2\ \upsilon$. It is universal among equalizing arrows, in the sense that for any object c and any cone like Figure 23 with vertex c instead of vertex $\mathrm{eq}(f_1, f_2)$ and with arrow $c \xrightarrow{g} a$ instead of $\mathrm{eq}(f_1, f_2) \xrightarrow{\upsilon} a$, there exists a unique arrow $c \xrightarrow{h_g} \mathrm{eq}(f_1, f_2)$ such that $\upsilon\ h_g = g$.

$$\begin{tikzcd}
& \mathrm{eq}(f_1, f_2) = \mathrm{Lim}\ S \arrow[dl, "\upsilon"'] \arrow[dr, "f_1\ \upsilon = f_2\ \upsilon"] & \\
a \arrow[rr, shift left, "f_1"] \arrow[rr, shift right, "f_2"'] & & b
\end{tikzcd}$$

Figure 23: The limit cone for $\mathrm{eq}(f_1, f_2)$.

In a category with zero arrows, we have $\mathrm{eq}(f_1, 0) = \ker f_1$ and $\mathrm{eq}(0, f_2) = \ker f_2$; this is clear from comparing Figure 23 with Figure 21. In the category R-**Mod**, we have $\mathrm{eq}(f_1, f_2) = \ker(f_1 - f_2) = \ker(f_2 - f_1)$, where $f_1 - f_2$ denotes the map $x \mapsto f_1\ x - f_2\ x$.

The **coequalizer** of f_1 and f_2, which we will denote $\mathrm{coeq}(f_1, f_2)$, is a colimit whose colimit cone has the form shown in Figure 23, after replacing eq with coeq, replacing Lim with Colim, and reversing all arrows and compositions. The universal property is the same as for an equalizer, after replacing eq with coeq and reversing all arrows and compositions.

In a category with zero arrows, we have $\mathrm{coeq}(f_1, 0) = \mathrm{coker}\ f_1$ and $\mathrm{coeq}(0, f_2) = \mathrm{coker}\ f_2$. In the category R-**Mod**, we have $\mathrm{coeq}(f_1, f_2) = \mathrm{coker}(f_1 - f_2) = \mathrm{coker}(f_2 - f_1)$.

Pullbacks and pushouts: A **pullback** in a category C is a limit in which the index category I of the functor $S\colon I \to C$ is described by the diagram $1 \to 2 \leftarrow 3$, plus the identity arrows. The limit cone has the commutative diagram shown in Figure 24. The universal property of the limit says that for any cone constructed from Figure 24 by replacing Lim S with c and υ with τ, there exists an arrow $c \xrightarrow{g_\tau} \mathrm{Lim}\ S$ such that for all $i \in I$, $\tau_i = \upsilon_i \circ g_\tau$.

$$\begin{tikzcd}
 & \mathrm{Lim}\ S \arrow[dl, "\upsilon_1"'] \arrow[d, "\upsilon_2"'] \arrow[dr, "\upsilon_3"] & \\
S\ 1 \arrow[r, "f_{12}"'] & S\ 2 & S\ 3 \arrow[l, "f_{32}"]
\end{tikzcd}$$

Figure 24: The limit cone for a pullback.

Note that the diagram in Figure 24 is completely determined by the four arrows υ_1, f_{12}, υ_3, and f_{32}, because we have $\upsilon_2 = f_{12}\ \upsilon_1 = f_{32}\ \upsilon_3$. For this reason, the limit diagram in Figure 24 is often written as a square, omitting the arrow υ_2.

A **pushout** in a category C is the dual concept of a pullback. It is a colimit in which the index category I is described by the diagram $1 \leftarrow 2 \to 3$, plus the identity arrows. The limit cone is as shown in Figure 24, after replacing Lim with Colim and reversing all arrows. The universal property is the same as for a pullback, after replacing Lim with Colim and reversing all arrows and compositions.

13. Functors to Set

In this section, we will consider functors from a category C to $\mathbf{Set}_V$ (the category of all sets contained in some set V), where V contains all the hom sets of C. If all C has small hom sets, we may take $V = U$ and write **Set** instead of $\mathbf{Set}_V$.

13.1. Covariant Functors

In this section we will consider a covariant functor $F\colon C \to \mathbf{Set}_V$.

We define the **Yoneda functor** $Y\colon C^{\mathrm{op}} \to \mathbf{Set}_V^C$ as follows:

1. Recall from § 6 that for any object a in C, $\hom_C(a, -) = C(a, -)$ is a functor from C to $\mathbf{Set}_V$, i.e., an object in the category $\mathbf{Set}_V^C$. The object function of this functor is $b \mapsto C(a, b)$. The arrow function is
$$f \mapsto C(a, f) = C(1_a, f).$$
The object function of Y takes each object a to the functor $Y\ a = C(a, -)$.
2. For any arrow $a \xrightarrow{f} b$, the rule $C(f, -)\ c = C(f, c)$ makes $C(f, -)$ into a natural transformation from the functor $C(b, -) = Y\ b$ to the functor $C(a, -) = Y\ a$. This natural transformation is an arrow in the category $\mathbf{Set}_V^C$.
The arrow function of Y takes each arrow $a \xrightarrow{f} b$ to the natural transformation $Y\ f = C(f, -)$.

The Yoneda functor is full and faithful.

Fix a functor $F\colon C \to \mathbf{Set}_V$ and an object a of C. Then $Y\ a$ and F are both functors from C to $\mathbf{Set}_V$, i.e., objects in the category $\mathbf{Set}_V^C$. Further, a natural transformation $\tau\colon Y\ a \stackrel{\cdot}{\to} F$ is an arrow in $\mathbf{Set}_V^C$, so it is an element of

$\mathbf{Set}_V^C(Y\ a, F)$. Observe the following:

- The component τ_a is an arrow (mapping of sets) from $Y\ a\ a = (Y\ a)\ a = C(a, a)$ to $F\ a$.
- The arrow 1_a in C is an element of $C(a, a)$, so $\tau_a\ 1_a$ is defined and is an element of $F\ a$.

We define the **Yoneda map** $y_a \colon \mathbf{Set}_V^C(Y\ a, F) \to F\ a$ as follows: $y_a\ \tau = \tau_a\ 1_a$. We will refer to $y_a\ \tau$ as the **Yoneda image** of the natural transformation $\tau \colon Y\ a \stackrel{\cdot}{\to} F$.

The Yoneda map is a bijection, so we have $\mathbf{Set}_V^C(Y\ a, F) \cong F\ a$ as sets. This statement is called the **Yoneda lemma** (proof omitted). It states that every natural transformation $\tau \colon Y\ a \stackrel{\cdot}{\to} F$ is completely determined by its Yoneda image $y_a\ \tau$.

The Yoneda map provides a natural isomorphism, in the following way:

1. Define the **natural transformation functor** $N \colon \mathbf{Set}_V^C \times C \to \mathbf{Set}_V$ as follows. The object function takes (F, a) to $\mathbf{Set}_V^C(Y\ a, F)$, the set of natural transformations from $Y\ a$ to F. The arrow function takes $(F \xrightarrow{\tau} G, a \xrightarrow{f} b)$ to the map $m \colon \mathbf{Set}_V^C(Y\ a, F) \to \mathbf{Set}_V^C(Y\ b, G)$ given by
$$m\ \sigma = m(\{Y\ a\ c \xrightarrow{\sigma_c} F\ c\}_{c \in C}) = \{Y\ b\ c \xrightarrow{Y\ f\ c} Y\ a\ c \xrightarrow{\sigma_c} F\ c \xrightarrow{\tau_c} G\ c\}_{c \in C}.$$
Here σ is a natural transformation, expressed as a family of components $\{\sigma_c\}_{c \in C}$. Similarly $m\ \sigma$ is a natural transformation, expressed as a family $\{(m\ \sigma)_c\}_{c \in C}$. As usual, the chained arrows indicate composition. In the "backwards" notation, the composition is $\tau_c \circ \sigma_c \circ (Y\ f\ c)$.
2. Define the **evaluation functor** $E \colon \mathbf{Set}_V^C \times C \to \mathbf{Set}_V$ as follows. The object function takes (F, a) to $F\ a$. The arrow function takes $(F \xrightarrow{\tau} G, a \xrightarrow{f} b)$ to $(G\ f) \circ \tau_a = (F\ f) \circ \tau_b$, where the equality holds by Figure 3.
3. Each Yoneda map y_a is the component at a of a natural isomorphism $y \colon N \stackrel{\cdot}{\to} E$.

A covariant functor $F \colon C \to \mathbf{Set}_V$ is **representable** if there exist an object a of C and a natural isomorphism $\phi \colon Y\ a = C(a, -) \stackrel{\cdot}{\to} F$. The pair (a, ϕ) is called a **representation** of F.

Fix categories C and D, a functor $F \colon C \to D$, and an object a of C. Let Y_C denote the Yoneda functor in the category C and Y_D denote the Yoneda functor in the category D. An arrow $s \xrightarrow{u} F\ a$ in D is a universal arrow from s to F (§ 9) if and only (1) if there exists a representation (a, ϕ) of the functor
$$D(s, F\ -) = D(s, -) \circ F = (Y_D\ s) \circ F,$$
i.e., a natural isomorphism $\phi \colon Y_C\ a \stackrel{\cdot}{\to} (Y_D\ s) \circ F$; and (2) u is the Yoneda image $y_a\ \phi = \phi_a\ 1_a$ (proof omitted).

13.2. Contravariant Functors

In this section we will consider a contravariant functor $F \colon C^{\mathrm{op}} \to \mathbf{Set}_V$.

The dual of the Yoneda functor is the functor $Y' \colon C \to \mathbf{Set}_V^{C^{\mathrm{op}}}$ defined as follows:

1. For any object a, $C(-, a)$ is a contravariant functor from C to $\mathbf{Set}_V$, i.e., an object in $\mathbf{Set}_V^{C^{\mathrm{op}}}$. The object function of Y' takes each object a to the functor $Y'\ a = C(-, a)$.
2. For any arrow $a \xrightarrow{f} b$, the rule $C(-, f)\ c = C(c, f)$ makes $C(-, f)$ into a natural transformation from the functor $C(-, a) = Y'\ a$ to the functor $C(-, b) = Y'\ b$. This natural transformation is an arrow in the category $\mathbf{Set}_V^{C^{\mathrm{op}}}$. The arrow function of Y' takes each arrow $a \xrightarrow{f} b$ to the natural transformation $C(-, f)$.

The functor Y' is full and faithful.

Fix a functor $F \colon C^{\mathrm{op}} \to \mathbf{Set}_V$ and an object a of C. Then $Y'\ a$ and F are both functors from C^{op} to $\mathbf{Set}_V$, i.e., objects in the category $\mathbf{Set}_V^{C^{\mathrm{op}}}$. Further, a natural transformation $\tau \colon Y'\ a \stackrel{\cdot}{\to} F$ is an arrow in $\mathbf{Set}_V^{C^{\mathrm{op}}}$, so it is an element of $\mathbf{Set}_V^{C^{\mathrm{op}}}(Y'\ a, F)$. We define the **dual Yoneda map** $y'_a \colon \mathbf{Set}_V^{C^{\mathrm{op}}}(Y'\ a, F) \to F\ a$ as follows: $y'_a\ \tau = \tau_a\ 1_a$. We will refer to $y'_a\ \tau$ as the **dual Yoneda image** of the natural transformation $\tau \colon Y'\ a \stackrel{\cdot}{\to} F$.

The dual Yoneda map is a bijection, so we have $\mathbf{Set}_V^{C^{\mathrm{op}}}(Y'\ a, F) \cong F\ a$ as sets. This statement is the dual of the Yoneda lemma (§ 13.1). It states that every natural transformation $\tau \colon Y'\ a \stackrel{\cdot}{\to} F$ is completely determined by its dual Yoneda image $y'_a\ \tau$.

The dual Yoneda map provides a natural isomorphism, in the following way:

1. Define the **natural transformation functor** $N: \mathbf{Set}_V^{C^{\mathrm{op}}} \times C \to \mathbf{Set}_V$ as follows. The object function takes (F, a) to $\mathbf{Set}_V^{C^{\mathrm{op}}}(Y'\ a, F)$, the set of natural transformations from $Y'\ a$ to F. The arrow function takes $(F \xrightarrow{\tau} G, a \xrightarrow{f} b)$ to the map $m: \mathbf{Set}_V^{C^{\mathrm{op}}}(Y'\ a, F) \to \mathbf{Set}_V^{C^{\mathrm{op}}}(Y'\ b, G)$ given by
$$m\ \sigma = m(\{F\ c \xrightarrow{\sigma_c} Y'\ a\ c\}_{c\ \in\ C}) = \{G\ c \xrightarrow{\tau_c} F_c \xrightarrow{\sigma_c} Y'\ a\ c \xrightarrow{Y'\ f\ c} Y'\ b\ c\}_{c\ \in\ C}.$$
2. Define the **evaluation functor** $E: \mathbf{Set}_V^{C} \times C \to \mathbf{Set}_V$ as follows. The object function takes (F, a) to $F\ a$. The arrow function takes $(F \xrightarrow{\tau} G, a \xrightarrow{f} b)$ to $\tau_a \circ (G\ f) = \tau_b \circ (F\ f)$, where the equality holds by Figure 3 with the vertical arrows reversed.
3. Each dual Yoneda map y'_a is the component at a of a natural isomorphism $y': N \dot{\to} E$.

A contravariant functor $F: C^{\mathrm{op}} \to \mathbf{Set}_V$ is **representable** if there exist an object a of C and a natural isomorphism $\phi: Y'\ a = C(-, a) \dot{\to} F$.

Fix categories C and D, a functor $F: C \to D$, and an object a of C. Let Y'_C denote the dual Yoneda functor in the category C and Y'_D denote the dual Yoneda functor in the category D. An arrow $F\ a \xrightarrow{u} t$ in D is a universal arrow from F to t (§ 9) if and only if (1) there exists a representation (a, ϕ) of the functor
$$D(F\ -, t) = D(-, t) \circ F = (Y'_D\ t) \circ F,$$
i.e., a natural isomorphism $\phi: Y'_C\ a \dot{\to} (Y'_D\ t) \circ F$; and (2) u is the dual Yoneda image $y'_a\ \phi = \phi_a\ 1_a$ (proof omitted).

14. Adjoint Functors

In this section we discuss adjoint functors. Adjoint functors (or adjunctions) are pairs of functors that are related in a special way. They appear frequently throughout mathematics.

14.1. Definition

Fix categories C_L and C_R and functors $F_L: C_L \to C_R$ and $F_R: C_R \to C_L$. Observe the following:

1. $C_R(F_L\ -, -) = C_R(-, -) \circ (F_L \times I_{C_R})$ is a bifunctor from $C_L \times C_R$ to $\mathbf{Set}_V$, contravariant in the first argument and covariant in the second argument.
2. $C_L(-, F_R\ -) = C_L(-, -) \circ (I_{C_L} \times F_R)$ is also a bifunctor from $C_L \times C_R$ to $\mathbf{Set}_V$, contravariant in the first argument and covariant in the second argument.

F_L and F_R are **adjoint functors**, and F_L is the **left adjoint**, and F_R is the **right adjoint**, if there is a natural isomorphism between $C_R(F_L\ -, -)$ and $C_L(-, F_R\ -)$. We usually write this natural isomorphism $C_R(F_L\ -, -) \cong C_L(-, F_R\ -)$, but we can also write $C_L(-, F_R\ -) \cong C_R(F_L\ -, -)$. The two expressions are equivalent. The significance of "left" and "right" lies in where F_L and F_R appear inside the expressions $C_R(F_L\ -, -)$ and $C_L(-, F_R\ -)$, not in the left and right sides of the isomorphism.

14.1.1. Right Adjuncts

We may express the natural isomorphism $C_R(F_L\ -, -) \cong C_L(-, F_R\ -)$ as a natural transformation
$$\phi: C_R(F_L\ -, -) \dot{\to} C_L(-, F_R\ -)$$
in which, for each object (a_L, a_R) in $C_L \times C_R$, the component
$$\phi_{(a_L, a_R)}: C_R(F_L\ a_L, a_R) \to C_L(a_L, F_R\ a_R)$$
is a bijection of sets. This bijection identifies arrows $F_L\ a_L \to a_R$ in C_R with arrows $a_L \to F_R\ a_R$ in C_L. Fix an arrow $f \in C_R(F_L\ a_L, a_R)$. The element $\phi_{(a_L, a_R)}\ f \in C_L(a_L, F_R\ a_R)$ is called the **right adjunct** of f.

A natural isomorphism of bifunctors is natural in each of its arguments (§ 11.1). Therefore, for each pair of objects (a'_L, a'_R) in $C_L \times C_R$, and for each pair of arrows $(f_L, f_R) \in C_L(a'_L, a_L) \times C_R(a_R, a'_R)$, the diagram shown in Figure 25 commutes. The top rectangle commutes because $\phi_{(-, a_R)}: C_R(F_L\ -, a_R) \dot{\to} C_L(-, F_R\ a_R)$ is a natural isomorphism. The bottom rectangle commutes because $\phi_{(a_L, -)}: C_R(F_L\ a_L, -) \dot{\to} C_L(a_L, F_R\ -)$ is a natural isomorphism.

$$\begin{tikzcd}
C_R(F_L\ a'_L, a_R) \arrow[r, "\phi_{(a'_L,a_R)}"] & C_L(a'_L, F_R\ a_R) \\
C_R(F_L\ a_L, a_R) \arrow[u, "C_R(F_L\ f_L, a_R) = f \mapsto f \circ (F_L\ f_L)"] \arrow[r, "\phi_{(a_L,a_R)}"] \arrow[d, "C_R(F_L\ a_L, f_R) = f \mapsto f_R \circ f"'] & C_L(a_L, F_R\ a_R) \arrow[u, "C_L(f_L, F_R\ a_R) = f \mapsto f \circ f_L"'] \arrow[d, "C_L(a_L, F_R\ f_R) = f \mapsto (F_R\ f_R) \circ f"] \\
C_R(F_L\ a_L, a'_R) \arrow[r, "\phi_{(a_L,a'_R)}"'] & C_L(a_L, F_R\ a'_R)
\end{tikzcd}$$

Figure 25: The commutative diagram for the natural isomorphism ϕ.

Equivalently, for all $f \in C_R(F_L\ a_L, a_R)$, the following identities hold:

$$\phi_{(a'_L,a_R)}(f \circ (F_L\ f_L)) = (\phi_{(a_L,a_R)}\ f) \circ f_L \quad \text{(top rectangle, naturality of } \phi_{(-,a_R)})$$

$$\phi_{(a_L,a'_R)}(f_R \circ f) = (F_R\ f_R) \circ (\phi_{(a_L,a_R)}\ f) \text{ (bottom rectangle, naturality of } \phi_{(a_L,-)})$$

14.1.2. Left Adjuncts

We may also express the natural isomorphism $C_R(F_L\ -,-) \cong C_L(-, F_R\ -)$ as a natural transformation

$$\psi = \phi^{-1} : C_L(-, F_R\ -) \dot{\to} C_R(F_L\ -,-)$$

in which, for each object (a_L, a_R) in $C_L \times C_R$, the component

$$\psi_{(a_L,a_R)} : C_L(a_L, F_R\ a_R) \to C_R(F_L\ a_L, a_R)$$

is a bijection of sets. Fix an arrow $f \in C_L(a_L, F_R\ a_R)$. The element $\psi_{(a_L,a_R)}\ f \in C_R(F_L\ a_L, a_R)$ is called the **left adjunct** of f. The dual of the diagram and of the identities shown for right adjuncts hold for left adjuncts.

14.2. The Unit and Counit

In this section, we will discuss the unit and counit, which are special natural transformations associated with each adjunction. Fix an adjunction A consisting of functors $F_L : C_L \to C_R$ and $F_R : C_R \to C_L$ with natural isomorphisms ϕ and $\psi = \phi^{-1}$ as in § 14.1.

14.2.1. The Unit

Observe that I_{C_L} (the identity functor on C_L) and the composite functor $F_R\ F_L$ are both functors from C_L to C_L. The **unit** associated with the adjunction A is the transformation η from I_{C_L} to $F_R\ F_L$ given by

$$\eta = \{\eta_{a_L} = \phi_{(a_L,F_L\ a_L)}\ 1_{F_L\ a_L}\}_{a_L \in C_L}.$$

To see that this definition makes sense, set $a_R = F_L\ a_L$ in the definition of $\phi_{(a_L,a_R)}$ and observe that

$$\phi_{(a_L,F_L\ a_L)} : C_R(F_L\ a_L, F_L\ a_L) \to C_L(a_L, F_R\ (F_L\ a_L))$$

is a bijection. Observe also that $1_{F_L\ a_L}$ is an arrow from $F_L\ a_L$ to itself, and so in the domain of this bijection. Therefore η_{a_L} is an arrow from $a_L = I_{C_L}\ a_L$ to $F_R\ (F_L\ a_L) = (F_R\ F_L)\ a_L$, as required for a transformation from I_{C_L} to $F_R\ F_L$.

We will now show that the unit η has the following properties:

1. η is a natural transformation $\eta : I_{C_L} \dot{\to} F_R\ F_L$.
2. For each object a_L in C_L, the arrow $a_L \xrightarrow{\eta_{a_L}} F_R\ (F_L\ a_L)$ is universal from a_L to F_R.
3. For every pair of objects (a_L, a_R) in $C_L \times C_R$ and every arrow $F_L\ a_L \xrightarrow{f} a_R$ in C_R, we have

$$\phi_{(a_L,a_R)}\ f = (F_R\ f) \circ \eta_{a_L}.$$

Property 1: For any arrow $a_L \xrightarrow{f} a'_L$ in C_L, we have

$$
\begin{aligned}
((F_R\ F_L)\ f) \circ \eta_{a_L} &= (F_R\ (F_L\ f)) \circ \eta_{a_L} \quad \text{(definition of composite functor)}\\
&= (F_R\ (F_L\ f)) \circ (\phi_{(a_L, F_L\ a_L)}\ 1_{F_L\ a_L}) \quad \text{(definition of } \eta_{a_L}\text{)}\\
&= \phi_{(a_L, F_L\ a'_L)}((F_L\ f) \circ 1_{F_L\ a_L}) \quad \text{(naturality of } \phi_{(a_L,-)}\text{)}\\
&= \phi_{(a_L, F_L\ a'_L)}(1_{F_L\ a'_L} \circ (F_L\ f)) \quad \text{(definition of identity arrow)}\\
&= (\phi_{(a'_L, F_L\ a'_L)}\ 1_{F_L\ a'_L}) \circ f \quad \text{(naturality of } \phi_{(-, F_L\ a'_L)}\text{)}\\
&= \eta_{a'_L} \circ f \quad \text{(definition of } \eta_{a'_L}\text{)}\\
&= \eta_{a'_L} \circ (I_{C_L}\ f) \quad \text{(definition of identity functor).}
\end{aligned}
$$

Property 2: By the remarks at the end of § 13.1, $s \xrightarrow{u} F\ a$ is universal to $F\colon C \to D$ if and only if (a) there exists a natural isomorphism $\phi_u\colon Y_C\ a \dot{\to} (Y_D\ s) \circ F$; and (b) u is the Yoneda image $y_a\ \phi_u = (\phi_u)_a\ 1_a$. Set $s = a_L$, $u = \eta_{a_L}$, $F = F_R$, and $a = F_L\ a_L$. Then $a_L \xrightarrow{\eta_{a_L}} F_R\ (F_L\ a_L)$ is universal to $F_R\colon C_R \to C_L$ if and only if (a) there exists a natural isomorphism

$$\phi_{\eta_{a_L}}\colon Y_{C_R}\ (F_L\ a_L) \dot{\to} (Y_{C_L}\ a_L) \circ F_R;$$

and (b) η_{a_L} is the Yoneda image $y_{F_L\ a_L}\ \phi_{\eta_{a_L}} = (\phi_{\eta_{a_L}})_{F_L\ a_L}\ 1_{F_L\ a_L}$.

a. Applying the definition of the Yoneda functor Y gives

$$\phi_{\eta_{a_L}}\colon C_R(F_L\ a_L, -) \dot{\to} C_L(a_L, F_R\ -).$$

Setting $\phi_{\eta_{a_L}} = \phi_{(a_L,-)}$ provides the required natural isomorphism.

b. By part (a),

$$(\phi_{\eta_{a_L}})_{F_L\ a_L}\ 1_{F_L\ a_L} = (\phi_{(a_L,-)})_{F_L\ a_L}\ 1_{F_L\ a_L}.$$

By the definition of the natural transformation $\phi_{(a_L,-)}$, this is $\phi_{(a_L, F_L\ a_L)}\ 1_{F_L\ a_L}$. This is exactly the definition of η_{a_L} given above.

Property 3: For every arrow $f \in C_R(F_L\ a_L, a_R)$, we have

$$
\begin{aligned}
\phi_{(a_L,a_R)}\ f &= \phi_{(a_L,a_R)}\ (f \circ 1_{F_L\ a_L}) \quad \text{(definition of identity arrow)}\\
&= (F_R\ f) \circ (\phi_{(a_L, F_L\ a_L)}\ 1_{F_L\ a_L}) \quad \text{(naturality of } \phi_{(a_L,-)}\text{)}\\
&= (F_R\ f) \circ \eta_{a_L} \quad \text{(definition of } \eta_{a_L}\text{).}
\end{aligned}
$$

14.2.2. The Counit

The counit is the dual construction to the unit. Observe that the composite functor $F_L\ F_R$ and I_{C_R} (the identity functor on C_R) are both functors from C_R to C_R. The **counit** associated with the adjunction A is the transformation ε from $F_L\ F_R$ to I_{C_R} given by

$$\varepsilon = \{\varepsilon_{a_R} = \psi_{(F_R\ a_R, a_R)}\ 1_{F_R\ a_R}\}_{a_R \in C_R}.$$

To see that this definition makes sense, set $a_L = F_R\ a_R$ in the definition of $\psi_{(a_L,a_R)}$ and observe that

$$\psi_{(F_R\ a_R, a_R)}\colon C_L(F_R\ a_R, F_R\ a_R) \to C_R(F_L\ (F_R\ a_R), a_R)$$

is a bijection. Observe also that $1_{F_R\ a_R}$ is an arrow from $F_R\ a_R$ to itself, and so in the domain of this bijection. Therefore ε_{a_R} is an arrow from $F_L\ (F_R\ a_R) = (F_L\ F_R)\ a_R$ to $a_R = I_{C_R}\ a_R$ as required for a transformation from $F_L\ F_R$ to I_{C_R}.

The counit ε has the following properties:

1. ε is a natural transformation $\varepsilon \colon F_L\ F_R \stackrel{\cdot}{\to} I_{C_R}$.
2. For each object a_R in C_R, the arrow $F_L\ (F_R\ a_R) \xrightarrow{\varepsilon_{a_R}} a_R$ is universal from F_L to a_R.
3. For every pair of objects (a_L, a_R) in $C_L \times C_R$ and every arrow $a_L \xrightarrow{f} F_R\ a_R$ in C_L, we have
$$\psi_{(a_L,a_R)}\ f = \varepsilon_{a_R} \circ (F_L\ f).$$

The proofs are dual to the proofs for the unit (§ 14.2.1).

14.2.3. The Composite Natural Transformations

By the observations at the end of § 7, we may form the natural transformations

$$\eta\ F_R \colon (I_{C_L}\ F_R = F_R) \stackrel{\cdot}{\to} F_R\ F_L\ F_R$$

and

$$F_R\ \varepsilon \colon F_R\ F_L\ F_R \stackrel{\cdot}{\to} (F_R\ I_{C_R} = F_R).$$

Then we may form the composite natural transformation

$$((F_R\ \varepsilon) \cdot (\eta\ F_R) \colon F_R \stackrel{\cdot}{\to} F_R) = a_R \mapsto ((F_R\ \varepsilon)\ a_R) \circ ((\eta\ F_R)\ a_R).$$

(§ 7). This composite is in fact the identity transformation $a_R \mapsto 1_{F_R\ a_R}$, because

$$\begin{aligned} ((F_R\ \varepsilon)\ a_R) \circ ((\eta\ F_R)\ a_R) &= (F_R\ \varepsilon_{a_R}) \circ \eta_{F_R\ a_R} \\ &= \phi_{(F_R\ a_R, a_R)}\ \varepsilon_{a_R} \quad \text{(property 3 of the unit)} \\ &= \phi_{(F_R\ a_R, a_R)}(\phi_{(F_R\ a_R, a_R)}{}^{-1}\ 1_{F_R\ a_R}) \quad \text{(definition of } \varepsilon_{a_R}\text{)} \\ &= 1_{F_R\ a_R}. \end{aligned}$$

Dually, we may form the composite natural transformation

$$((\varepsilon\ F_L) \cdot (F_L\ \eta) \colon F_L \stackrel{\cdot}{\to} F_L) = a_L \mapsto ((\varepsilon\ F_L)\ a_L) \circ ((F_L\ \eta)\ a_L),$$

and this composite is the identity transformation $a_L \mapsto 1_{F_L\ a_L}$.

14.3. Example

In this section we work through a detailed example of adjoint functors. For many more examples, see, e.g., [Mac Lane 1998].

The categories and functors: Set $C_L = C_R = \mathbf{Set}$. Let hom denote $\hom_{\mathbf{Set}}$. Fix a set B in **Set**.

1. Let $- \times B$ be the covariant functor that takes a set A in **Set** to the set $A \times B$ and takes a function $f \colon A \to A'$ in **Set** to the function $f \times 1_B = (a, b) \mapsto (f(a), b) \colon A \times B \to A' \times B$,
2. As usual, let $F_R = \hom(B, -)$ be the covariant functor that takes a set C in **Set** to the set $\hom(B, C)$ and takes a function $f \colon C \to C'$ in **Set** to the function $\hom(B, f) = g \mapsto g \circ f \colon \hom(B, C) \to \hom(B, C')$.

We will demonstrate that setting $F_L = - \times B$ and $F_R = \hom(B, -)$ yields an adjunction. This adjunction is an expression in category theory of the relationship called **currying** in computer science. For a similar adjunction consisting of the functors $- \otimes_R B$ and $\hom_{R\text{–}\mathbf{Mod}}(B, -)$ in the category R–**Mod**, see my paper *Definitions for Commutative Algebra*.

The bijection ϕ: For each pair (A, C) in $\mathbf{Set} \times \mathbf{Set}$, the bijection ϕ has type

$$\phi_{(A,C)} \colon \hom(A \times B, C) \to \hom(A, \hom(B, C)).$$

For each function $f \colon A \times B \to C$, let

$$\phi_{(A,C)}\ f = a \mapsto (b \mapsto f\ (a, b)).$$

Then ϕ has the correct type, and it has an inverse

$$\phi_{(A,C)}{}^{-1}\ g = \psi_{(A,C)}\ g = (a, b) \mapsto g\ a\ b$$

defined for all $g\colon A \to (B \to C)$, so it is a bijection.

The naturality of ϕ: Let $a_L = A$ and $a_R = C$, and fix functions $f_L\colon A' \to A$ and $f_R\colon C \to C'$. We must show that both rectangles of Figure 25 commute, for any $f \in \hom(A \times B, C)$.

Bottom rectangle: We have

$$\phi_{(A,C')}\ (f_R \circ f) = a \mapsto (b \mapsto (f_R \circ f)\ (a, b)) = a \mapsto (b \mapsto f_R\ f\ (a, b)).$$

On the other hand, we have

$$\begin{aligned}(F_R\ f_R) \circ (\phi_{(A,C)}\ f) &= \hom(B, f_R) \circ (\phi_{(A,C)}\ f)\\ &= (g \mapsto f_R \circ g) \circ (a \mapsto (b \mapsto f\ (a, b)))\\ &= a \mapsto (f_R \circ (b \mapsto f\ (a, b)))\\ &= a \mapsto (b \mapsto f_R\ f\ (a, b)).\end{aligned}$$

Top rectangle: We have

$$\begin{aligned}\phi_{(A',C)}\ (f \circ (F_L\ f_L)) &= \phi_{(A',C)}\ (f \circ (f_L \times B))\\ &= \phi_{(A',C)}\ (f \circ ((a', b) \mapsto (f_L\ a', b)))\\ &= \phi_{(A',C)}\ ((a', b) \mapsto f\ (f_L\ a', b))\\ &= a' \mapsto (b \mapsto f\ (f_L\ a', b)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\phi_{(A,C)}\ f) \circ f_L &= (a \mapsto (b \mapsto f\ (a, b))) \circ f_L\\ &= a' \mapsto (b \mapsto f\ (f_L\ a', b)).\end{aligned}$$

The unit: For each set A, η_A is an arrow from A to $\hom(B, A \times B)$. It is the function

$$\phi_{(A,A\times B)}\ 1_{(A\times B)} = a \mapsto (b \mapsto 1_{(A\times B)}\ (a, b)) = a \mapsto (b \mapsto (a, b)).$$

As expected, the arrow

$$A \xrightarrow{\eta_A} \hom(B, A \times B) = \hom(B, -)\ (A \times B)$$

is universal from A to $\hom(B, -)$. Indeed, if $f\colon A \to \hom(B, S)$ is any mapping of sets, then

$$g_f = (a, b) \mapsto f\ a\ b$$

is the unique function such that $\hom(B, g_f) \circ \eta_A = f$. Also as expected, for every pair of sets (A, C) in $\mathbf{Set} \times \mathbf{Set}$ and every arrow $A \times B \xrightarrow{f} C$ in **Set**, we have

$$\phi_{(A,C)}\ f = (F_R\ f) \circ \eta_A = \hom(B, f) \circ \eta_A.$$

Indeed,

$$\begin{aligned}\hom(B, f) \circ \eta_A &= (g \mapsto f \circ g) \circ (a \mapsto (b \mapsto (a, b)))\\ &= a \mapsto (f \circ (b \mapsto (a, b)))\\ &= a \mapsto (b \mapsto f\ (a, b))\end{aligned}$$

$$= \phi_{(A,C)}\ f.$$

The counit: For each set C in Set, ε_C is an arrow from $\hom(B,C) \times B$ to C. It is the function

$$\psi_{(\hom(B,C),C)}\ 1_{\hom(B,C)} = (f,b) \mapsto (1_{\hom(B,C)}\ f)\ b = (f,b) \mapsto f\ b.$$

As expected, the arrow

$$(- \times B)\ \hom(B,C) = \hom(B,C) \times B \xrightarrow{\varepsilon_C} C$$

is universal from $- \times B$ to C. Indeed, if $f\colon S \times B \to C$ is any mapping of sets, then

$$g_f = s \mapsto (b \mapsto f\ (s,b))$$

is the unique function such that $\varepsilon_A \circ (g_f \times B) = f$. Also as expected, for every pair of sets (A,C) in **Set** $\times$ **Set** and every function $A \xrightarrow{f} \hom(B,C)$ in **Set**, we have

$$\psi_{(A,C)}\ f = \varepsilon_C \circ (f \times B).$$

Indeed,

$$\varepsilon_C \circ (f \times B) = ((f,b) \mapsto f\ b) \circ ((a,b) \mapsto (f\ a, b))$$

$$= (a,b) \mapsto f\ a\ b$$

$$= \psi_{(A,C)}\ f.$$

The composite natural transformations: As expected, for every set C in **Set**, we have

$$\hom(B, \varepsilon_C) \circ \eta_{\hom(B,C)} = (g \mapsto \varepsilon_C \circ g) \circ (f \mapsto (b \mapsto (f,b)))$$

$$= f \mapsto (b \mapsto \varepsilon_C\ (f,b))$$

$$= f \mapsto (b \mapsto f\ b)$$

$$= f \mapsto f$$

$$= 1_{\hom(B,C)}.$$

Also as expected, for every set A in **Set**, we have

$$\varepsilon_{A \times B} \circ (\eta_A \times B) = ((f,b) \mapsto f\ b) \circ ((a,b) \mapsto (b \mapsto (a,b), b))$$

$$= (a,b) \mapsto ((b \mapsto (a,b))\ b)$$

$$= (a,b) \mapsto (a,b)$$

$$= 1_{A \times B}.$$

15. Conclusion

This document has defined some of the fundamental concepts of category theory. For more advanced material, see, e.g., [Mac Lane 1998]. If you master the concepts in this document, you should be in a good position to learn more. The discussion in [Mac Lane 1998] skips many details; the examples in this paper should suggest how to fill them in. For a discussion of monads applied to computer science, see my paper *Monads, Categories, and Computation*.

References

Mac Lane, Saunders. *Categories for the Working Mathematician.* Second Edition. Springer Verlag, 1998.

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